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30	30. МАТЕМАТИЧЕСКОЕ МОДЕЛИРОВАНИЕ ПРОЦЕССОВ В СИСТЕМАХ С ОБРАТНОЙ СВЯЗЬЮ

RINGS AND ALGEBRAS OBTAINED FROM OTHER ALGEBRAIC STRUCTURES

BY

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1. Current text-books of algebra contain the following problem : to obtain new algebraic structures endowed or not with order and (or) topology, from old ones. So are finite direct products and quotient structures.

In present our short Note we obtain rings and algebras from old ones in a way generally enough. The finite direct products appear as special cases. In a similar way we have obtained groups in [1] and [2].

By a ring we mean an additive commutative group with a distributive product and by an algebra over a field we mean a vectorial space over a field with a product distributive with respect to the sum and which satisfies $\alpha(ab) = (\alpha a)b = a(\alpha b)$, α in the field, a and b in algebra.

2. **Theorem 1.** *Let R_k be rings, $k = \overline{1, 3}$ and let I_k be right ideal of R_k $k = \overline{1, 3}$. Let $f_k : I_k \rightarrow R_{k+1}$ be morphisms of additive groups. $k = \overline{1, 3}$, $R_4 = R_1$. In $I_1 \times I_2 \times I_3$ define the sum $(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$ and the product; we fix a' and a'' in R_1 , b' and b'' in R_2 and c' and c'' in R_3 ; $(x_1, x_2, x_3)(y_1, y_2, y_3) = [(x_1 y_1) a' + (x_1 f_3(y_3) + y_1 f_3(x_3)) a'', (x_2 y_2) b' + (x_2 f_1(y_1) + y_2 f_1(x_1)) b'', (x_3 y_3) c' + (x_3 f_2(y_2) + y_3 f_2(x_2)) c'']$. Then $I_1 \times I_2 \times I_3$ is a ring with respect to the two operations.*

Proof. We must prove the distributivity only.

$$\begin{aligned} \text{The right distributivity. } I &= (x_1, x_2, x_3)(z_1, z_2, z_3) + (y_1, y_2, y_3)(z_1, z_2, z_3) = \\ &= [(x_1 z_1) a' + (x_1 f_3(z_3) + z_1 f_3(x_3)) a'', (x_2 z_2) b' + (x_2 f_1(z_1) + z_2 f_1(x_1)) b'', \\ & \quad (x_3 z_3) c' + (x_3 f_2(z_2) + z_3 f_2(x_2)) c''] + [(y_1 z_1) a' + (y_1 f_3(z_3) + z_1 f_3(y_3)) a'', \\ & \quad (y_2 z_2) b' + (y_2 f_1(z_1) + z_2 f_1(y_1)) b'', (y_3 z_3) c' + (y_3 f_2(z_2) + z_3 f_2(y_2)) c''] = \\ &= [(x_1 z_1 + y_1 z_1) a' + (x_1 f_3(z_3) + z_1 f_3(x_3) + y_1 f_3(z_3) + z_1 f_3(y_3)) a'', (x_2 z_2 + y_2 z_2) b' + \\ & \quad + (x_2 f_1(z_1) + z_2 f_1(x_1) + y_2 f_1(z_1) + z_2 f_1(y_1)) b'', (x_3 z_3 + y_3 z_3) c' + \\ & \quad + (x_3 f_2(z_2) + z_3 f_2(x_2) + y_3 f_2(z_2) + z_3 f_2(y_2)) c'']. \end{aligned}$$

$$\begin{aligned} II &= (x_1 + y_1, x_2 + y_2, x_3 + y_3)(z_1, z_2, z_3) = [((x_1 + y_1) z_1) a' + ((x_1 + y_1) f_3(z_3) + \\ & \quad + z_1 f_3(x_3 + y_3)) a'', ((x_2 + y_2) z_2) b' + ((x_2 + y_2) f_1(z_1) + z_2 f_1(x_1 + y_1)) b'', \\ & \quad ((x_3 + y_3) z_3) c' + ((x_3 + y_3) f_2(z_2) + z_3 f_2(x_2 + y_2)) c'']. \end{aligned}$$

One sees that $I = II$.

$$\begin{aligned} \text{The left distributivity. } I' &= (z_1, z_2, z_3)(x_1, x_2, x_3) + (z_1, z_2, z_3)(y_1, y_2, y_3) = \\ &= [(z_1x_1)a' + (z_1f_3(x_3) + x_1f_3(z_3))a'', (z_2x_2)b' + (z_2f_1(x_1) + x_2f_1(z_1))b'', \\ &\quad (z_3x_3)c' + (z_3f_2(x_2) + x_3f_2(z_2))c''] + [(z_1y_1)a' + (z_1f_3(y_3) + y_1f_3(z_3))a'', \\ &\quad (z_2y_2)b' + (z_2f_1(y_1) + y_2f_1(z_1))b'', (z_3y_3)c' + (z_3f_2(y_2) + y_3f_2(z_2))c''] = \\ &= [(z_1x_1 + z_1y_1)a' + (z_1f_3(x_3) + x_1f_3(z_3) + z_1f_3(y_3) + y_1f_3(z_3))a'', \\ &\quad (z_2x_2 + z_2y_2)b' + (z_2f_1(x_1) + x_2f_1(z_1) + z_2f_1(y_1) + y_2f_1(z_1))b'', \\ &\quad (z_3x_3 + z_3y_3)c' + (z_3f_2(x_2) + x_3f_2(z_2) + z_3f_2(y_2) + y_3f_2(z_2))c'']. \end{aligned}$$

$$\begin{aligned} II' &= (z_1, z_2, z_3)(x_1 + y_1, x_2 + y_2, x_3 + y_3) = [(z_1(x_1 + y_1))a' + (z_1f_3(x_3 + y_3) + \\ &\quad + (x_1 + y_1)f_3(z_3))a'', (z_2(x_2 + y_2))b' + (z_2f_1(x_1 + y_1) + (x_2 + y_2)f_1(z_1))b'', \\ &\quad + (z_3(x_3 + y_3))c' + (z_3f_2(x_2 + y_2) + (x_3 + y_3)f_2(z_2))c'']. \end{aligned}$$

One sees that $I' = II'$.

Commentary. 1. One sees that the Theorem 1 is true for n rings, n ideals, and n morphisms, $n \geq 2$.

2. One sees that some or all $a', b', c', a'', b'', c''$ can be deleted.

3. When $I_k = R_k$, $k = \overline{1, n}$, $f_k = 0$ (or the "seconds" are zero), $k = \overline{1, n}$, $R_{n+1} = R_1$ and all "primes" are deleted, one obtains the finite direct product of rings.

4. For $I_1 = I_2 = R_1 = R_2 = Z$ — the ring of integers, $n = 2$ (in 1), $f_2 = 0$ (or $a'' = 0$), $R_3 = R_1 = Z$, $a' = b'' = 1$, f_1 — the identity, one obtains the problem 19 ([3], p. 80).

5. There is a similar Theorem for the case when I_k is left ideal of R_k , $k = \overline{1, n}$ (see [1]).

6. When I_k is a bilateral ideal of R_k , $k = \overline{1, n}$ (see [1]), "the prime and second elements" and " $f_k(t)$ ", $k = \overline{1, n}$, can be fixed everywhere and we have similar Theorems.

7. The ring $I_1 \times I_2 \times I_3$ is not associative. Same the others.

3. **Theorem 2.** Let A_k be algebras over the field C , $k = \overline{1, 3}$, and let E_k be right ideal (of algebra) of A_k , $k = \overline{1, 3}$. Let $f_k: E_k \rightarrow A_{k+1}$ be morphisms of vector spaces. $k = \overline{1, 3}$, $A_4 = A_1$. In $E_1 \times E_2 \times E_3$ define the sum and product as in Theorem 1. Also in $E_1 \times E_2 \times E_3$ define external operation $\alpha(x_1, x_2, x_3) = (\alpha x_1, \alpha x_2, \alpha x_3)$, $\alpha \in C$. Then $E_1 \times E_2 \times E_3$ is algebra with respect to the above three operations.

Proof. By virtue of the Theorem 1, we must verify only

$$\begin{aligned} \alpha[(x_1, x_2, x_3)(y_1, y_2, y_3)] &= (\alpha(x_1, x_2, x_3))(y_1, y_2, y_3) = \\ &= (x_1, x_2, x_3)(\alpha(y_1, y_2, y_3)). \end{aligned}$$

$$\begin{aligned} I'' &= \alpha[(x_1, x_2, x_3)(y_1, y_2, y_3)] = \alpha[(x_1y_1)a' + (x_1f_3(y_3) + y_1f_3(x_3))a'', \\ &\quad (x_2y_2)b' + (x_2f_1(y_1) + y_2f_1(x_1))b'', (x_3y_3)c' + (x_3f_2(y_2) + y_3f_2(x_2))c'']. \end{aligned}$$

$$\begin{aligned} II'' &= (\alpha x_1, \alpha x_2, \alpha x_3)(y_1, y_2, y_3) = [((\alpha x_1)y_1)a' + ((\alpha x_1)f_3(y_3) + y_1f_3(\alpha x_3))a'', \\ &\quad ((\alpha x_2)y_2)b' + ((\alpha x_2)f_1(y_1) + y_2f_1(\alpha x_1))b'', ((\alpha x_3)y_3)c' + ((\alpha x_3)f_2(y_2) + y_3f_2(\alpha x_2))c'']. \end{aligned}$$

$$\begin{aligned}
 III'' = (x_1, x_2, x_3)(\alpha y_1, \alpha y_2, \alpha y_3) = & [(x_1(\alpha y_1))a' + (x_1 f_3(\alpha y_3) + (\alpha y_1)f_3(x_3))a'', \\
 & (x_2(\alpha y_2))b' + (x_2 f_1(\alpha y_1) + (\alpha y_2)f_1(x_1))b'', \\
 & (x_3(\alpha y_3))c' + (x_3 f_2(\alpha y_2) + (\alpha y_3)f_2(x_2))c''].
 \end{aligned}$$

One sees that $I'' = II'' = III''$.

Commentary similar to the previous one.

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INELE ȘI ALGEBRE OBTINUTE DIN ALTE STRUCTURI ALGEBRICE

(Rezumat)

Din ideale de inele (algebre), cu ajutorul morfismelor de grup (de spațiu vectorial), se obțin inele (algebre) noi. Produsele directe finite de inele (algebre) apar ca un caz particular al procedurii destul de general prezentată în această lucrare.

DIRICHLET-FINITE HARMONIC FUNCTIONS ON STANDARD H -CONES (I)

BY

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In [4] we have introduced a notion of Dirichlet integral on a standard H -cones of functions, S . We suppose that S is autodual; then S can be identified with its dual S^* . In the presence of the axiom D and some other additional axioms, we have defined a Dirichlet integral, denoted by $\delta_{[f^*, g]}$ for any f, g which are differences of bounded, naturally continuous super-harmonic functions

$$\delta_{[f, g]} = \frac{1}{2} \{ f\sigma(g) + g\sigma(f) - \sigma(fg) \},$$

where $\sigma(p)$ means the unique non-negative measure for which $p(x) = \int g(x, y) d\sigma(p)(y)$ if p is a potential. $\sigma(f)$ is the measure which represents the potential component from the Riesz decomposition of f . $g(x, y)$ is the symmetric Green function defined in [5]. We have proved that for any p, q , bounded nearly continuous elements of S ,

$$(1) \quad \delta_{[p, q]}(1) = \int_X g(x, y) d\sigma(p)(x) d\sigma(q)(y),$$

and for any bounded substractible elements h , with finite Dirichlet integral

$$(2) \quad \delta_{[h, p]}(1) = 0.$$

We want to establish that (1) and (2) are valid for locally bounded elements of S .

Let $S = S^{**}$ be a standard H -cone of functions on a saturated set X , which is autodual relative to an H -morphism denoted by $\varphi: S \rightarrow S^*$. We suppose that $1 \in S_H$, where S_H is the set of all substractible elements of S . S satisfies the axioms A and D. X is locally connected with the respect to the natural topology.

For any $U \subset X$, naturally open, we denote by δ^U the gradient measure on the standard autodual H -cone $\varphi(U)$. If $U_1 \subset U_2$ are naturally open, we have

$$\delta_{f|_{U_1}}^{U_1} = f|_{U_1} \sigma_{U_1}(f|_{U_1}) - \frac{1}{2} \sigma_{U_1}(f|_{U_1}^2) = (f \sigma_{U_2}(f))|_{U_1} - \frac{1}{2} (\sigma_{U_2}(f))|_{U_1} = (\delta_f^{U_2})|_{U_1}.$$

Let $\mathcal{R} = \{f: X \rightarrow R \mid \text{such that } \exists (U_i)_{i \in I}, \text{ naturally open, } X = \bigcup_{i \in I} U_i, f|_{U_i} \in [\varphi_M(U_i)]\}$, where $\varphi_M(U)$ is the set of all bounded elements of $\varphi(U)$.

We recall from [2] the notation $S_\Delta = \{s \in S \mid [s, s] < \infty\}$. We know that $S_\Delta \subset S_p$, where S_p is the set of all nearly continuous elements of S .

Proposition 1. For any $p, q \in S_\Delta \cap \mathcal{R}$, naturally continuous, we have $\delta_p(1) < \infty$, $\delta_q(1) < \infty$, $\delta_{[p, q]}(1) < \infty$ and $\delta_{[p, q]}(1) = [p, q]$.

Proof. There exists $(U_n)_{n \in \mathbb{N}}$, U_n naturally open, with $U_n \subset U_{n+1}$, such that $X = \bigcup_{n \in \mathbb{N}} U_n$, and $p|_{U_n}, q|_{U_n}$ are bounded. We choose $\varphi_n \in \mathcal{O}(X)$, such that $\varphi_n = 1$ on $\bar{U}_n$ and $\varphi_n = 0$ on $X \setminus U_{n+1}$. We denote $p_n = R(\varphi_n p) \leq p$, $q_n = R(\varphi_n q) \leq q$. It results that $p_n \uparrow p$ and $q_n \uparrow q$. From ([4], Prop. 3.3.3) for any $m, n \in \mathbb{N}$, we deduce

$$(3) \quad \delta_{[p_n, q_m]}(1) = [p_n, q_m].$$

Because $p_n|_{U_n} = p|_{U_n}$, we have $\delta_p(U_n) = \delta_{p|_{U_n}}^n(U_n) = \delta_{p_n}(U_n) \leq \delta_{p_n}(1) = [p_n, p_n] \leq [p, p] < \infty$. We deduce also $\delta_q(U_n) \leq \|q\| < \infty$. Passing to supremum we find $\delta_p(1) < \infty$, $\delta_q(1) < \infty$ and from the inequality $\delta_{[p, q]}(1) \leq \delta_p^{1/2}(1) \delta_q^{1/2}(1)$, we get $\delta_{[p, q]}(1) < \infty$.

For any $m \in \mathbb{N}$ and $n, n' \in \mathbb{N}$, with $n' > n$,

$$\begin{aligned} |\delta_{[p-p, q_m]}(U_{n'})| &= |\delta_{[p_n-p_n', q_m]}(U_{n'})| \leq \delta_{p_n-p_n'}^{1/2}(U_{n'}) \delta_{q_m}^{1/2}(U_{n'}) \leq \delta_{p_n-p_n'}^{1/2}(1) \delta_{q_m}^{1/2}(1) = \\ &= \|p_n - p_{n'}\|^{1/2} \|q_m\|^{1/2} \rightarrow \|p_n - p\|^{1/2} \|q_m\|^{1/2} \quad (n' \rightarrow \infty). \end{aligned}$$

Hence $|\delta_{[p_n-p, q_m]}(1)| \leq \|p_n - p\|^{1/2} \|q_m\|^{1/2}$ and $\lim_{n \rightarrow \infty} \delta_{[p_n-p, q_m]}(1) = 0$. It results

$$\delta_{[p, q_m]}(1) = \lim_{n \rightarrow \infty} \delta_{[p_n, q_m]}(1) = \lim_{n \rightarrow \infty} [p_n, q_m] = [p, q_m].$$

Let be $m' > m$. Then

$$|\delta_{[p, q_m-q]}(U_{m'})| = |\delta_{[p, q_m-q_m']}|(U_{m'}) \leq \delta_p^{1/2}(1) \|q_m - q_m'\|^{1/2} \rightarrow \delta_p^{1/2}(1) \|q_m - q\|^{1/2} \quad (m' \rightarrow \infty).$$

Hence $|\delta_{[p, q_m-q]}(1)| \leq \delta_p^{1/2}(1) \|q_m - q\|$. It results

$$(4) \quad \lim_{n \rightarrow \infty} \delta_{[p, q_m-q]}(1) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \delta_{[p, q_m]}(1) = \delta_{[p, q]}(1).$$

Passing to limit in (3), we get

$$\delta_{[p, q]}(1) = \lim_{m \rightarrow \infty} \delta_{[p, q_m]}(1) = \lim_{m \rightarrow \infty} [p, q_m] = [p, q].$$

Lemma 2. Let be $p \in S \cap \mathcal{R}$, naturally continuous, and $(U_n)_{n \in \mathbb{N}}$ naturally open, such that $\bar{U}_n \subset U_{n+1}$, $\bigcup_{n \in \mathbb{N}} U_n = X$, $p|_{U_n}$ bounded. If $B_n = B^{\beta(X \setminus U_n)}$, then $\lim_{n \rightarrow \infty} \delta_{B_n p}(U_n) = 0$.

Proof. We have $p|_{U_n} = p - B_n p + B_n p|_{U_n}$. Let $p_n = p - B_n p$. Because p is a naturally potential, we have $\bigwedge_{n \in \mathbb{N}} B_n p = 0$, hence $\lim_{n \rightarrow \infty} p_n = p$. From (Prop. 1.3.8 [4]), $B_n p|_{U_n}$ is subtractible in $\varphi(U_n)$ and from (Corollary 1.3.8 [4]) we get

$$\delta_{[B_n p|_{U_n}, p_n]}^n(U_n) = 0,$$

where δ^n is $\delta|_{U_n}$. Then

$$\begin{aligned}
\delta_{B_n p|U_n}^n(U_n) &= \delta_{p|U_n - p_n}^n(U_n) = \delta_{p|U_n}^n(U_n) - \delta_{p_n}^n(U_n) - 2\delta_{[p|U_n - p_n, p_n]}^n(U_n) = \\
&= \delta_{p|U_n}^n(U_n) - \delta_{p_n}^n(U_n) - 2\delta_{[B_n p|U_n, p_n]}^n(U_n) = \delta_{p|U_n}^n(U_n) - \delta_{p_n}^n(U_n) = \\
&= \delta_p(U_n) - \int_X p_n d\sigma_{U_n}(p_n) = \delta_p(U_n) - \int_{U_n} p_n d\sigma(p)|_{U_n} = \\
&= \delta_p(U_n) - \int_X \chi_{U_n} p_n d\sigma(p) \leq \delta_p(1) - \int_X \chi_{U_n} p_n d\sigma(p).
\end{aligned}$$

From Lebesgue convergence theorem, the statement of the Lemma is true.

Proposition 3. If $h \in \mathcal{H}_D$ and $p \in S_\Delta \cap \mathcal{Q}$, then

$$\delta_{[h, p]}(1) = 0.$$

Proof. Let be $(U_n)_{n \in \mathbf{N}}$ from the definition of $\mathcal{Q}$, such that $\bar{U}_n \subset U_{n+1}$ and $p|_{U_n}, h|_{U_n}$ are bounded. For p_n defined in Lemma 2, we have

$$\delta_{[h|U_n, p_n]}^n(U_n) = 0,$$

$$\begin{aligned}
\delta_{[h|U_n, p_n - p|U_n]}^n(U_n) &\leq \delta_h^{1/2}(U_n) \delta_{p_n - p|U_n}^{1/2}(U_n) = \delta_h^{1/2}(U_n) \delta_{B_n p|U_n}^{1/2}(U_n) \leq \\
&\leq \delta_h^{1/2}(1) \delta_{B_n p|U_n}^{1/2}(U_n) \rightarrow 0, \quad n \rightarrow \infty.
\end{aligned}$$

Hence

$$0 = \lim_{n \rightarrow \infty} \delta_{[h|U_n, p_n]}^n(U_n) = \lim_{n \rightarrow \infty} \delta_{[h|U_n, p|U_n]}^n(U_n) = \lim_{n \rightarrow \infty} \delta_{[h, p]}^n(U_n) = \delta_{[h, p]}(1).$$

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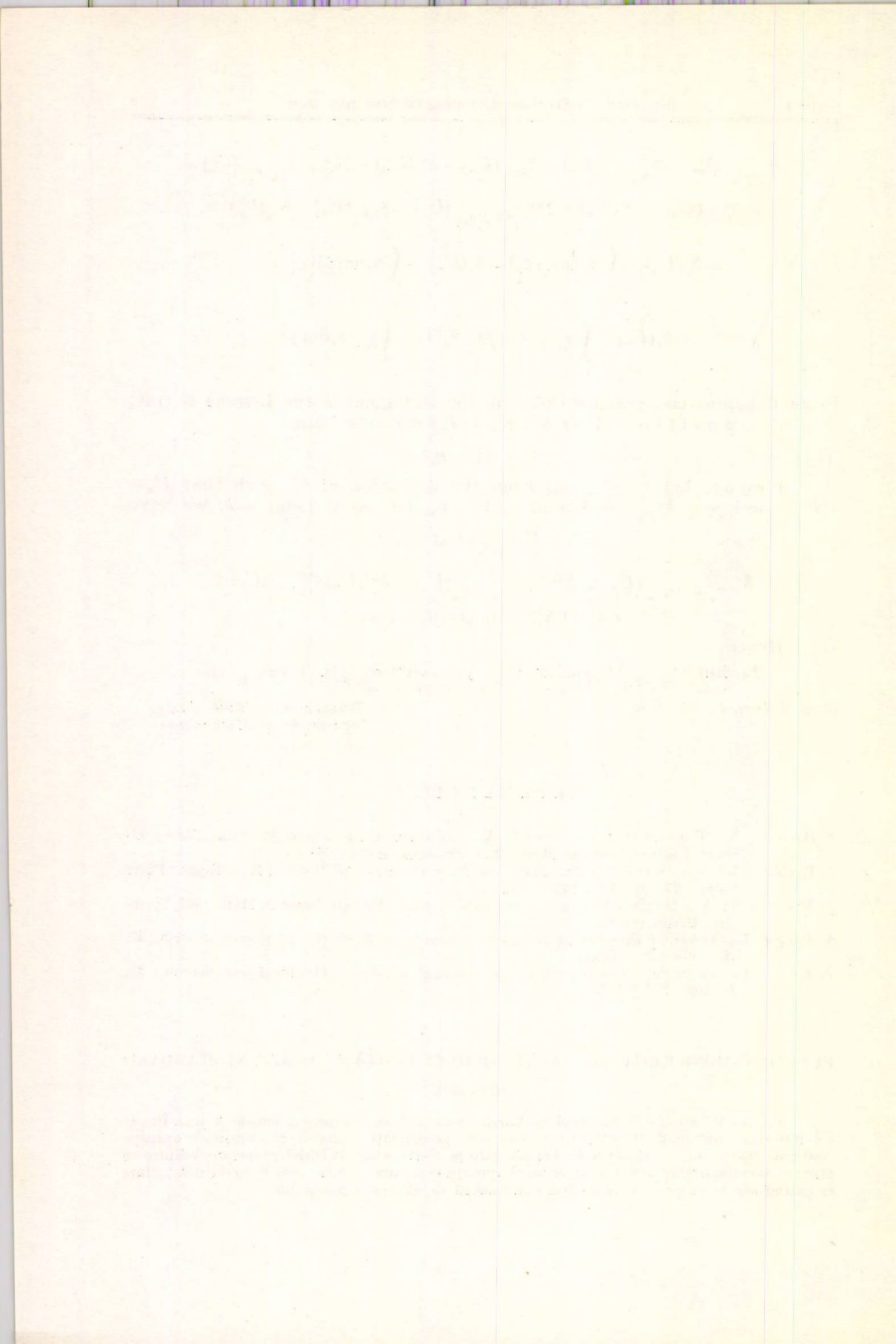
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FUNCTII ARMONICE CU INTEGRALĂ DIRICHLET FINITĂ ÎN H-CONURI STANDARD

(Rezumat)

Fie $S = S^{**}$ un H -con standard de funcții, autodual, ce satisface axiomele A și D. În [4] s-a introdus o integrală Dirichlet cu următoarele proprietăți: coincide pe elementele aproape continue, mărginite, cu valoarea formei biliniare pe S , iar integrala Dirichlet reciprocă dintre un element substractibil mărginit și un element aproape continuu mărginit este 0. În această lucrare se extind aceste proprietăți la o clasă convenabilă de elemente nemărginite.



CONNEXIONS LINÉAIRES ATTACHÉES À UNE CONNEXION SPINORIELLE (II)

PAR

ELENA PÎȘCAȘU

1. Systèmes de matrices Dirac. Soit V_4 la variété espace-temps de la relativité générale, $g_{\alpha\beta}(\alpha, \beta, \gamma, \dots = 0, 1, 2, 3)$, le tenseur métrique de la variété et $p_\alpha = (p_{ab})$, $(a, b, c, \dots = 1, 2, 3, 4)$ un système de matrices Dirac, donc matrices complexes, qui vérifient la relation

$$(1) \quad p_\alpha p_\beta + p_\beta p_\alpha = -2g_{\alpha\beta}e,$$

e — la matrice unité d'ordre 4.

Le spin-tenseur p_{ab}^a défini par les matrices Dirac p_α on nomme spin-tenseur fondamental de l'espace V_4 .

On suppose que sur V_4 nous avons une structure presque-complexe définie par le tenseur φ_α^β , donc

$$(2) \quad \varphi_\alpha^\rho \varphi_\rho^\beta = -\delta_\alpha^\beta.$$

Nous avons encore

$$(3) \quad \varphi_{\alpha\beta} = \varphi_\alpha^\rho g_{\rho\beta},$$

$$(4) \quad \varphi_{\alpha\beta} = -\varphi_{\beta\alpha}.$$

On considère les matrices h_α définies de

$$(5) \quad h_\alpha = F_\alpha^\beta(\lambda, \mu) p_\beta,$$

où $F_\alpha^\beta(\lambda, \mu)$ est donnée par

$$(6) \quad F_\alpha^\beta(\lambda, \mu) = \lambda \varphi_\alpha^\beta + \mu \delta_\alpha^\beta, \quad \forall \lambda, \mu \in \mathbb{R}$$

et, évidemment, satisfait les relations

$$(7) \quad \begin{cases} F_\alpha^\beta(\lambda, \mu) F_\beta^\rho(-\lambda, \mu) = \delta_\alpha^\rho, \\ F_\alpha^\beta(\lambda, \mu) F_\beta^\rho(\mu, \lambda) = \varphi_\alpha^\rho. \end{cases}$$

Dans [5] on démontre que h_α forment un système de matrices Dirac si et seulement si $\lambda^2 + \mu^2 = 1$,

2. Connexions linéaires attachées à une connexion spinorielle. Si ω est une connexion linéaire arbitraire sur V_4 dans [1] on détermine toutes les connexions spinorielles δ ainsi que la dérivée covariante du spin-tenseur fondamental p_{ab}^a par rapport au couple (ω, σ) soit nulle. En particulier, si ω est une connexion riemannienne σ avec la propriété énoncé avant, elle est unique et coïncide avec la connexion spinorielle Lichnerowicz.

Dans cette Note, nous nous proposons, que en partant des connexions spinorielles σ déterminées en [1], de trouver toutes les connexions linéaires Γ ainsi que la dérivée covariante du spin-tenseur h_{ab}^a par rapport au couple (Γ, σ') soit nulle. De

$$(8) \quad D_\alpha h_{\beta b}^a = 0,$$

on obtient

$$(9) \quad \Gamma_{\beta\gamma}^\alpha(\lambda, \mu) = \omega_{\beta\gamma}^\alpha + \lambda^2 \varphi_\beta^\theta \nabla_\gamma \varphi_\theta^\alpha + \lambda \mu \nabla_\gamma \varphi_\beta^\alpha, \quad \text{avec} \quad \lambda^2 + \mu^2 = 1,$$

∇ étant le symbole de la dérivation covariante par rapport à la connexion linéaire ω .

Nous obtenons ainsi, une famille de connexions linéaires $G(\lambda, \mu)$ des éléments $\Gamma(\lambda, \mu)$ données par (9).

Remarque. De (9) résulte

$$(10) \quad \Gamma(-\lambda, -\mu) = \Gamma(\lambda, \mu); \quad \Gamma(-\lambda, \mu) = \Gamma(\lambda, -\mu).$$

En utilisant le même procédé comme dans [5], on montre que sur $G(\lambda, \mu)$ peut être introduite une opération par rapport de la quelle $G(\lambda, \mu)$ a une structure de groupe commutatif, avec $\Gamma(0, 1) = \omega$ — élément neutre et $\Gamma(-\lambda, \mu)$ — élément symétrique de l'élément $\Gamma(\lambda, \mu)$.

3. Propriétés de la connexion $\Gamma(\lambda, \mu)$. Si D est le symbole de la dérivation covariante par rapport à Γ , nous avons

$$(11) \quad D_\gamma g_{\alpha\beta} = F_{\alpha}^{\rho}(\lambda, -\mu) F_{\beta}^{\theta}(\lambda, -\mu) \nabla_\gamma g_{\rho\theta}$$

et

$$(12) \quad \nabla_\gamma g_{\rho\tau} = F_{\rho}^{\alpha}(-\lambda, -\mu) F_{\tau}^{\beta}(-\lambda, -\mu) D_\gamma g_{\alpha\beta},$$

d'où résulte le

Théorème 1. *La condition nécessaire et suffisante pour que Γ soit une connexion métrique est que ω soit une connexion métrique.*

Définition Nous disons que le couple de connexions (ω, Γ) $\forall \Gamma \in G(\lambda, \mu)$ est F -compatible avec la structure presque-complexe donnée par φ si en transportant par parallélisme sur une courbe quelconque de V_4 un vecteur arbitraire v^α relatif à la connexion $\Gamma(\lambda, \mu)$, le vecteur $u^\alpha = F_{\beta}^{\alpha}(\lambda, \mu) v^\beta$ se transporte parallèlement relatif à la connexion ω sur la même courbe.

Théorème 2. *Le couple (ω, Γ) est F -compatible avec la structure presque-complexe donnée par φ .*

Vraiment, soit le vecteur v^α et nous définissons le vecteur

$$(13) \quad u^\alpha = F_{\beta}^{\alpha}(\lambda, \mu) v^\beta.$$

Nous avons

$$(14) \quad \frac{dv^\alpha}{dt} + \Gamma_{\beta\gamma}^\alpha(\lambda, \mu) v^\beta \frac{dx^\gamma}{dt} = 0.$$

De (9), (13), et (14) résulte

$$(15) \quad \frac{du^\alpha}{dt} + \omega_{\beta\gamma}^\alpha u^\beta \frac{dx^\gamma}{dt} = 0.$$

4. Les invariants du groupe $G(\lambda, \mu)$. Les tenseurs de torsion et de courbure de la connexion Γ sont donnés par

$$(16) \quad T_{\beta\gamma}^\alpha(\lambda, \mu) = t_{\beta\gamma}^\alpha + \lambda^2(\varphi_\beta^\theta \nabla_\gamma \varphi_\theta^\alpha - \varphi_\gamma^\theta \nabla_\beta \varphi_\theta^\alpha) + \lambda \mu (\nabla_\gamma \varphi_\beta^\alpha - \nabla_\beta \varphi_\gamma^\alpha),$$

$$(17) \quad R_{\beta\epsilon\tau}^\alpha(\lambda, \mu) = \mu^2 r_{\beta\epsilon\tau}^\alpha + \lambda \mu \varphi_\beta^\rho \varphi_{\rho\epsilon\tau}^\alpha - \lambda \mu \varphi_\epsilon^\rho \varphi_{\rho\beta\tau}^\alpha - \lambda^2 \varphi_\beta^\theta \varphi_\epsilon^\rho \varphi_{\rho\theta\tau}^\alpha,$$

ou $t_{\beta\gamma}^\alpha$ et $r_{\beta\epsilon\tau}^\alpha$ sont respectivement, le tenseur de torsion et le tenseur de courbure de la connexion ω .

À l'aide de ces tenseurs, on peut obtenir les invariants suivants du groupe $G(\lambda, \mu)$:

$$(18) \quad I_{1\beta\gamma}^\alpha = T_{\beta\gamma}^\alpha(\lambda, \mu) - F_\theta^\alpha(\lambda^2, -\lambda\mu)(\nabla_\beta \varphi_\gamma^\theta - \nabla_\gamma \varphi_\beta^\theta) = t_{\beta\gamma}^\alpha,$$

$$(19) \quad I_{2\beta\gamma}^\alpha = T_{\beta\gamma}^\alpha(\lambda, \mu) - F_\theta^\alpha(\lambda^2, -\lambda\mu - 1)(\nabla_\beta \varphi_\gamma^\theta - \nabla_\gamma \varphi_\beta^\theta) = t_{\beta\gamma}^\alpha + \nabla_\beta \varphi_\gamma^\alpha - \nabla_\gamma \varphi_\beta^\alpha$$

$$(20) \quad I_{3\beta\gamma}^\alpha = T_{\beta\gamma}^\alpha(\lambda, \mu) - \varphi_\beta^\rho \varphi_\gamma^\tau T_{\rho\tau}^\alpha(\lambda, \mu) - \varphi_\rho^\alpha \varphi_\beta^\epsilon T_{\epsilon\gamma}^\rho(\lambda, \mu) - \varphi_\rho^\alpha \varphi_\gamma^\epsilon T_{\beta\epsilon}^\rho(\lambda, \mu) = \\ = t_{\beta\gamma}^\alpha - \varphi_\beta^\rho \varphi_\gamma^\tau t_{\rho\tau}^\alpha - \varphi_\rho^\alpha \varphi_\beta^\epsilon t_{\epsilon\gamma}^\rho - \varphi_\rho^\alpha \varphi_\gamma^\epsilon t_{\beta\epsilon}^\rho,$$

$$(21) \quad J_{1\beta\epsilon\tau}^\alpha = F_\rho^\alpha(\lambda, \mu) F_\beta^\nu(-\lambda, \mu) R_{\nu\epsilon\tau}^\rho(\lambda, \mu) = r_{\beta\epsilon\tau}^\alpha,$$

$$(22) \quad J_{2\beta\epsilon\tau}^\alpha = F_\rho^\alpha(\mu, -\lambda) F_\beta^\nu(\mu, \lambda) R_{\nu\epsilon\tau}^\rho(\lambda, \mu) = \varphi_\rho^\alpha \varphi_\beta^\nu r_{\nu\epsilon\tau}^\rho,$$

$$(23) \quad J_{3\beta\epsilon\tau}^\alpha = \varphi_\theta^\alpha R_{\beta\epsilon\tau}^\theta(\lambda, \mu) + \varphi_\beta^\theta R_{\theta\epsilon\tau}^\alpha(\lambda, \mu) = \varphi_\theta^\alpha r_{\beta\epsilon\tau}^\theta + \varphi_\beta^\theta r_{\theta\epsilon\tau}^\alpha,$$

$$(24) \quad J_{4\beta\epsilon\tau}^\alpha = R_{\beta\epsilon\tau}^\alpha(\lambda, \mu) - \varphi_\beta^\theta \varphi_\epsilon^\alpha R_{\theta\epsilon\tau}^\rho(\lambda, \mu) = r_{\beta\epsilon\tau}^\alpha - \varphi_\beta^\theta \varphi_\epsilon^\alpha r_{\theta\epsilon\tau}^\rho.$$

5. Les propriétés des invariants I et J . Si $\Phi_1, \Phi_2, \Phi_3, \Phi_4$ sont les opérateurs Obata construits à l'aide du tenseur φ , on obtient les propriétés suivantes des invariants I et J :

$$(25) \quad \Phi_{1\beta\tau}^{\rho\alpha} I_{3\rho\gamma}^\tau = 0, \quad \Phi_{3\beta\gamma}^{\eta\epsilon} I_{3\eta\epsilon}^\alpha = 0,$$

$$(26) \quad I_{3\beta\gamma}^\alpha = 2\Phi_{4\beta\gamma}^{\rho\tau} t_{\rho\tau}^\alpha + 2\Phi_{1\gamma\rho}^{\epsilon\alpha} t_{\rho\epsilon}^\alpha + 2\Phi_{2\beta\rho}^{\epsilon\alpha} t_{\rho\epsilon}^\alpha,$$

$$(27) \quad \Phi_{1\gamma\alpha}^{\beta\epsilon} J_{2\beta\rho\tau}^\alpha = \Phi_{1\gamma\alpha}^{\beta\epsilon} r_{\beta\rho\tau}^\alpha,$$

$$(28) \quad J_{2\beta\epsilon\tau}^\alpha + r_{\beta\epsilon\tau}^\alpha = 2\Phi_{1\beta\rho}^{\theta\alpha} r_{\theta\epsilon\tau}^\rho,$$

$$(29) \quad \Phi_{1\gamma\alpha}^{\beta\epsilon} J_{3\beta\rho\tau}^\alpha = 0,$$

$$(30) \quad J_{4\beta\epsilon\tau}^\alpha = 2\Phi_{2\beta\eta}^{\rho\alpha} r_{\rho\epsilon\tau}^\eta.$$

Une interprétation géométrique des invariants $I_1, J_1; I_2, J_2$ est donnée par les

Théorème 3. *Il y a une seule connexion du groupe $G(\lambda, \mu)$, $\Gamma(0, 1)$ qui a comme tenseur de torsion l'invariant I_1 et comme tenseur de courbure l'invariant J_1 .*

Théorème 4. *Il y a une seule connexion du groupe $G(\lambda, \mu)$, $\Gamma(1, 0)$ qui a comme tenseur de torsion l'invariant I_2 et comme tenseur de courbure l'invariant J_2 .*

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CONEXIUNI LINIARE ATAȘATE UNEI CONEXIUNI SPINORIALE (II)

(Rezumat)

Se consideră varietatea spațiu-timp din relativitatea generală și se construiește pe această varietate un sistem de matrice Dirac $h_\alpha = (h_{\alpha b}^a)$. Considerind conexiunea spinorială δ dată în [1] se determină toate conexiunile liniare Γ astfel ca derivata covariantă a spin-tensorului $h_{\alpha b}^a$ în raport cu perechea (Γ, σ) să fie nulă. Mulțimea acestor conexiuni are structură de grup comutativ. Se determină o serie de invarianți ai acestui grup și se studiază unele proprietăți ale lor.

PIPELINED PROCESSING OF TASKS WITH DETERMINISTIC OR STOCHASTIC EXECUTION TIMES

BY

OCTAV BRUDARU and RADU SILION

1. Introduction. Let us consider the acyclic digraph $G=(V, A)$, where the set of vertices $V=\{1, 2, \dots, n\}$ denotes the set of tasks and the set of arcs indicates the precedence constraints imposed to the processing of tasks. We suppose that $\{V_a, V_s\}$ is a partition of V so that for each $i \in V_a$, $td(i)$ is a positive integer representing the execution time of i -th task, while for each $j \in V_s$, $ts(j)$ is a random variable taking a finite number of positive integer values, giving the execution time of j -th task. The random variables $ts(j)$, $j \in V_s$, are assumed to be independently. If $W \subset V$ we define the positive integer $T_a(W)$ and the random variable $T_s(W)$ as follows

$$T_a(W) = \sum_{x \in W \cap V_a} td(x) \quad \text{and} \quad T_s(W) = \sum_{x \in W \cap V_s} ts(x).$$

If $p_0 \in (0, 1]$ and C is a positive integer called cycle time, then let S be the set of all partitions $Q = \{Q_1, \dots, Q_m\}$ of V satisfying

- (1) $\forall (x, y) \in A, \quad x \in Q_j, \quad y \in Q_k \Rightarrow j \leq k,$
- (2) $\begin{cases} Q_j \cap V_a \neq \emptyset, Q_j \cap V_s \neq \emptyset \Rightarrow T_a(Q_j) \leq C; \\ Q_j \cap V_a = \emptyset, Q_j \cap V_s \neq \emptyset \Rightarrow P(T_s(Q_j) \leq C) \geq p_0; \\ Q_j \cap V_a \neq \emptyset, Q_j \cap V_s \neq \emptyset \Rightarrow T_a(Q_j) \leq C \text{ and } P(T_s(Q_j) \leq C - T_a(Q_j)) \geq p_0. \end{cases}$

The pipelining with deterministic or stochastic execution times (PDSET) problem consists in finding $Q^* \in S$ so that

$$(3) \quad |Q^*| = \min\{|Q| \mid Q \in S\}.$$

Each $Q \in S$ is a solution to the problem, while $Q^* \in S$ defined by (3) is an optimal solution. Further we suppose that

$$(4) \quad td(i) \leq C, \quad \forall i \in V_a$$

and

$$(5) \quad P(ts(j) \leq C) \geq p_0, \quad \forall j \in V_s.$$

This implies that if $Q_j = \{x_j\}$, $j=1, \dots, n$, where the permutation $(x_1, \dots, x_n)$ of V sorts the elements of V in topological order ([4], p. 312), then $S \neq \emptyset$ because $\{Q_1, \dots, Q_n\} \in S$.

A partition $\{Q_1, \dots, Q_m\} \in S$ permits to design a linear array of processors in order to process the tasks $1, \dots, n$ in a manner which extensively employs pipelining processing. A such array requires the similar processors $P_1, \dots, P_m$. The processor P_j is programmed to execute the tasks belonging to Q_j in a serial fashion in accordance to the precedence constraints. The data flow passes from P_j to P_{j+1} , $j=1, \dots, m-1$. At each C units of time it is expected that a new batch is completely processed and leaves the array, while another batch enters P_1 . The number of processors must be minimized.

We note that the problems concerning the sizing of data buffers used to connect the processors will not be discussed here.

Let us remark that if $V_s = \emptyset$ we obtain the assembly line balancing problem with deterministic execution time, formulated in [2]. A simple proof of the NP-completeness of the associated decision problem is given in [1]. Thus, the PDSET problem is NP-hard.

For $V_d = \emptyset$, a preference order dynamic programming procedure is proposed in [5] and critically analysed in [6]. In this paper it is presented a branch and bound approach of the PDSET problem. Also we are giving a heuristic having a polynomial time complexity.

2. Preliminary Results. A permutation $x = (x_1, \dots, x_n)$ of V is called admissible if $(x_i, x_j) \in A$ implies $i < j$. Let AP_G be the set of all admissible permutations of V . If G is acyclic then $AP_G \neq \emptyset$. If $x = (x_1, \dots, x_n) \in AP_G$ then let us consider the digraph $G_x = (V, A_x)$, where $A_x = \{(x_j, x_{j+1}) / j=1, \dots, n-1\}$. Let S_x be the set of solutions of PDSET instance obtained by replacing the digraph G by G_x . The following statement results at once.

Proposition 1. *In the above conditions it follows that*

$$S = \bigcup_{x \in AP_G} S_x.$$

A consequence of this proposition is

Proposition 2. *If $|Q_x^*| = \min\{|Q_x| / Q_x \in S_x\}$ for each $x \in AP_G$ then $\min\{|Q| / Q \in S\} = \min\{|Q_x^*| / x \in AP_G\}$.*

If $x = (x_1, \dots, x_n) \in AP_G$ we note by $m(x; n)$ the value of variable m after the execution of the following algorithm.

Algorithm SEQ

Step 0: Take $m := 1$ and $R_1 := \emptyset$.

Step 1: For $i=1$ to n do:

(1.1) If $x_i \in V_d$ then compute $t: T_d(R_m \cup \{x_i\})$ and $p := P(T_s(R_m) \leq C - t)$. If $t \leq C$ and $p \geq p_0$ then $R_m := R_m \cup \{x_i\}$, otherwise take $m := m + 1$ and $R_m := \{x_i\}$.

(1.2) If $x_i \in V_s$ then compute $p := P(T_s(R_m \cup \{x_i\}) \leq C - T_d(R_m))$. If $p \geq p_0$ then $R_m := R_m \cup \{x_i\}$, otherwise take $m := m + 1$ and $R_m := \{x_i\}$.

Proposition 3. If $R = \{R_1, \dots, R_m\}$ is offered by the Algorithm SEQ then $R \in S_x$ and $m(x; n) = \min\{|Q|/|Q \in S_x|\}$.

Proof. It is obvious that $R \in S_x$. Let us suppose that the optimal solution of the PDSET instance associated with x is $W = \{W_1, \dots, W_q\}$. It is clear that if $|W_j| = c_j$ then $W_j = \{x_{a(j)+h}/h = 1, \dots, c_j\}$, for $j = 1, \dots, q$. By using the step (1.1) or the step (1.2) of the previous algorithm it results that if $f(j)$ is the subscript of the last task assigned to R_j , then $a(j+1) \leq f(j)$ for $j = 1, \dots, q-1$, where $a(1) = 0$ and $a(k) = c_1 + \dots + c_{k-1}$, $k = 1, \dots, q$. By applying again the step (1.1) or the step (1.2) it results that $m(x; n) \leq q$. Thus R is optimal.

Let us note that the execution of SEQ takes $O(gn)$ time, where g is an upper bound of the time needed to compute the required probabilities.

As a consequence of Propositions 1...3 we have

Proposition 4. In the above conditions it results that

$$\min\{|Q|/|Q \in S|\} = \min\{m(x; n)/x \in AP_G\}.$$

Let us note that if G contains a hamiltonian path then $|AP_G| = 1$ and PDSET problem is solvable in polynomial time.

3. A Branch and Bound Approach. Further, we shall construct a tree T associated with S and we shall define an estimate function in order to drive the exploring of T .

3.1. The branching rules. Let $T = (N, E)$ be the tree, where N is the set of vertices and E is the set of arcs. If $z \in N$ and $W \subset V$ then $\text{succ}(z) = \{t/t \in N, (z, t) \in E\}$, $G(W) = (V_W, A_W)$ is the digraph with $V_W = V \setminus W$ and $A_W = \{(a, b)/a, b \in V_W, (a, b) \in A\}$, and $I(W) = \{a/a \in V_W, \{b/b \in V_W, (b, a) \in A_W\} = \emptyset\}$. The tree T and a labeling function $f: N \rightarrow V \cup \{0\}$ are constructed as it follows. Let us denote by N_k the k -th level of T , where $k \in \{0, \dots, n\}$.

Rule 1. If z_0 is the root of T we put $f(z_0) = 0$.

Rule 2. If $z \in N_k$ with $k \in \{0, 1, \dots, n-1\}$ and $D(z) = [z_0, z_1, \dots, z_k = z]$ is the path from z_0 to z in T , we construct the set $B = I(\{f(z_1), \dots, f(z_k)\})$. If $B = \{x_1, \dots, x_r\}$ then $\text{succ}(z) = \{t_1, \dots, t_r\}$ and $f(t_j) = x_j, j = 1, \dots, r$.

Rule 3. If $z \in N_n$ then $\text{succ}(z) = \emptyset$.

Now, if $z \in N_k$, $k \in \{1, \dots, n\}$, and $D(z) = [z_0, \dots, z_k = z]$ we note by $AP(z)$ the set of all admissible permutations having the prefix of length k equal to $(f(z_1), \dots, f(z_k))$. It holds

Proposition 5. If T is constructed by applying rules 1...3 then for each $k \in \{0, \dots, n-1\}$ and $z \in N_k$

(i) $AP(z') \cap AP(z'') = \emptyset, \forall z', z'' \in \text{succ}(z)$,

(ii) $AP(z) = \bigcup_{z' \in \text{succ}(z)} AP(z')$,

and

(iii) $AP_G = \bigcup_{z \in N_n} AP(z)$.

The proof is immediate and it uses essentially the fact that G is acyclic.

Let us remark that Rule 2 can be efficiently implemented if a data structure like for the topological sorting ([4], p. 310) is associated with each vertex z able to be branched.

3.2. *The estimation function.* We define the estimate function $e: N \setminus \{z_0\} \rightarrow \{1, \dots, n\}$ as it follows. If $k \in \{1, \dots, n\}$, $z \in N_k$ and $D(z) = [z_0, \dots, z_k = z]$ then we take $x = (x_1, \dots, x_k)$, where $x_j = f(z_j)$, $j = 1, \dots, k$. Let $m(x; k)$ be the value of variable m after applying of Algorithm SEQ to x , where n was replaced by k . The following statement holds.

Proposition 6. *In the above conditions, if $e(z) = m(x; k)$ then*

- (i) $e(z) \leq m(y; n)$, $\forall y \in AP(z)$, $\forall z \in N_k$, with $k \in \{1, \dots, n-1\}$,
- (ii) $e(z) = m(y; n)$, $AP(z) = \{y\}$, $\forall z \in N_n$.

Propositions 5 and 6 show that by applying the least cost search strategy ([3], pp. 8–22), in order to find a terminal vertex z^* of T such that $e(z^*) \leq e(z)$ for each active vertex z of T , then an optimal solution PDSET problem can be found.

Let us remark that $e(z)$ can be computed recursively as it follows. If $e(z)$ is already computed, Q_m is the last class associated with z and $z' \in \text{succ}(z)$, then by putting $f(z')$ and Q_m instead of x_i and R_m respectively, and by applying either step (1.1) or step (1.2) from Algorithm SEQ, we have that $e(z')$ is equal to the resulted value of m .

4. A Heuristic Method. Further we shall use the notations introduced in the previous sections. The proposed heuristic is given below.

Algorithm HEURIST

Step 1: Construct $I(\emptyset)$, choice an arbitrary task $x_1^* \in I(\emptyset)$, take $Q_1 = \{x_1^*\}$ and $m := 1$.

Step 2: For $k = 2, 3, \dots, n$ do:

(2.1) Construct $B: I(\{x_1^*, \dots, x_{k-1}^*\})$, take $s_k := |B|$, $B = \{y_1, \dots, y_{s_k}\}$ and $c := 0$.

(2.2) For $i := 1, \dots, n$ do:

(2.2.1) If $y_i \in V_a$ compute $t := T_a(Q_m \cup \{y_i\})$ and $p := P(T_s(Q_m) \leq C - t)$.

If $t \leq C$ and $p \geq p_0$ then $x_k := y_i$, $Q_m := Q_m \cup \{y_i\}$ and $c := 1$, otherwise pass to next i .

(2.2.2) If $y_i \in V_s$ and $P(T_s(Q_m \cup \{y_i\})) \leq C - T_a(Q_m) \geq p_0$ then take $x_k := y_i$, $Q_m := Q_m \cup \{y_i\}$ and $c := 1$, otherwise pass to next i .

(2.3) If $c = 0$ then choice an arbitrary task $x_k^* \in B$, take $m := m + 1$, $Q_m := \{x_k^*\}$ and pass to next k .

Proposition 7. *If G is acyclic, conditions (4)–(5) are satisfied and $Q = \{Q_1, \dots, Q_m\}$ is given by Algorithm HEURIST then*

- (i) $Q \in S$,
- (ii) *if G contains a hamiltonian path then Q is optimal.*

Sketch of proof. (i) The sets $I(\emptyset)$ and $I(\{x_1^*, \dots, x_k^*\})$, $k = 2, \dots, n-1$ are nonempty because G is acyclic. If a class Q_m cannot be completed in step (2.2.1) or (2.2.2) then a new class can be created in step (2.3), because (4) and (5) are satisfied. Because each task x_k^* cannot be assigned before, the assignment of its direct predecessors, it results that Q verifies (1). The assertion (ii) results from Proposition 4.

Second, the Algorithm HEURIST is applied and it is obtained the optimal solution $\{\{1, 2\}, \{5, 3, 4, 6\}\}$.

6. Practical Applications. The paper originates from the designing of a system dedicated to acquisition and processing of experimental data coming from nonrepetitive tests. The obtained results have been used for reducing the number of microprocessors and for programming each of them in order to execute the tasks forming the associated class.

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PROCESARE DE TIP „PIPELINE” A SARCINILOR CU TIMPI DE EXECUȚIE DETERMINIȘTI SAU STOCHASTICI

(Rezumat)

Se prezintă o tehnică „branch and bound” și o metodă euristică pentru proiectarea unui tablou liniar de procesare a unei mulțimi de sarcini având timpii de execuție determinați sau stochastici, execuția fiind supusă la restricții de precedență.

ON THE EQUATION $f(y')y''=2ay$

BY

ADRIAN CORDUNEANU

In the most important particular case, when $f(u)=(1+u^2)^{-3/2}$, to integrate this equation is equivalent to find all the curves of the plane (x, y) along which the curvature is proportional with the ordinate y . This situation occurs in some technical problems. E. K a m k e ([1], p. 584) has shown that the corresponding differential equation may be reduced to the form

$$(1) \quad y' = \pm \frac{1}{2ay^2 - C} \sqrt{4 - (2ay^2 - C)^2}, \quad C = \text{const.}$$

Unfortunately, the formula giving the general solution of the above equation (taking $x = x(y)$) is enough complicated. However, we shall deduce some properties of the particular solutions of the considered equation

$$(2) \quad \frac{y''}{(1+y'^2)^{3/2}} = 2ay, \quad a = \text{const.} > 0$$

satisfying prescribed initial values $y(0)$ and $y'(0)$, following a somewhat similar method. We mention that another question related to (2), namely the boundary value problem

$$(3) \quad y'' + \lambda y(1+y'^2)^{3/2} = 0, \quad y(-b) = y(b) = 0.$$

was considered by R. C a c c i o p p o l i [2]. He proved that (3) has a unique positive solution $y=y(x)$, for every λ contained in some positive determined interval. In the general case, we suppose that for the equation

$$(4) \quad f(y')y'' = 2ay, \quad a = \text{const.} > 0,$$

where $f=f(u)$ is defined and positive on the whole real axis, the existence and the uniqueness of the solutions satisfying given initial values $y(0)$ and $y'(0)$ are guaranteed. Then, we can state some elementary remarks related to the Eq. (4).

Remark 1. If $f=f(u)$ is an even function and $y(x)$ is a particular solution of (4), then $-y(x)$ is also solution.

Remark 2. If $f=f(u)$ is an even function and $y=y(x)$ is a solution of (4) defined on some interval I , then $z(x)=y(-x)$ is also solution, defined on $I' =$ the symmetric of I with respect to the origin.

Remark 3. If $G'(u)=f(u)u$ and $G(u)$ is bounded, then every solution of (4) is bounded.

Indeed, from $f(y')y'y''=2ayy'$ we obtain $G(y')=ay^2+\text{const.}$, which implies that $y=y(x)$ is bounded.

Remark 4. Assume that $f=f(u)$ is an even function and that $y(x)$ with $y'(x_0)=0$ is a solution of (4), defined on $[x_0, x_0+L)$. Then, prolonging $y(x)$ by parity with respect to the point x_0 , we obtain a solution of (4) defined on the interval (x_0-L, x_0+L) .

To justify this assertion, we make use of the relation $y(x_0-x)=y(x_0+x)$ for $x \in (0, L)$, which implies $y(x)=y(2x_0-x)$ for $x \in (x_0-L, x_0)$. We easily obtain $f(y'(x))y''(x)=2ay(x)$ for $x \in (x_0-L, x_0)$; the condition $y'(x_0)=0$ assures that the prolonged function has the first and the second derivatives for $x=x_0$.

Remark 5. Let $y=y(x)$ be a solution of (4), defined on the maximal finite interval (x_m, x_M) and denote $T=x_M-x_m$. Prolonging this function by periodicity with the period T , we obtain solutions of (4) defined on the intervals (x_m+kT, x_M+kT) , where $k=\text{integer}$.

The case $f(u)=(1+u^2)^{-3/2}$. Multiplying the corresponding equation (2) by y' and integrating on $[0, x]$, we obtain

$$(5) \quad \alpha - (1+y'^2)^{-1/2} = a(y^2 - \eta^2),$$

where $\alpha = (1+y'^2(0))^{-1/2} \in (0, 1]$ and $\eta = y(0)$ are constants for a given solution. From (5), it follows

$$(6) \quad y' = \pm \frac{\sqrt{1 - [\alpha - a(y^2 - \eta^2)]^2}}{\alpha - a(y^2 - \eta^2)}, \quad x = \pm \int_{\eta}^y \frac{\alpha - a(y^2 - \eta^2)}{\sqrt{1 - [\alpha - a(y^2 - \eta^2)]^2}} dy$$

and now we distinguish more situations, depending of the initial data η and α :

(i) $y(0)=\eta \geq 1/\sqrt{a}$, $y'(0)>0$. To obtain the corresponding solution of (2), it must take the sign $+$ in the second formula (6). The interval of variation for y is determined, of course, by the conditions that in (6) the denominator be $\neq 0$ and the expression under the radical be nonnegative; in this case, this interval is $[(\eta^2 + (\alpha - 1)/a)^{1/2}, (\eta^2 + \alpha/a)^{1/2}]$. Denoting

$$(7) \quad y_0 = \sqrt{\eta^2 + \frac{\alpha - 1}{a}}, \quad x_0 = \int_{\eta}^{y_0} \frac{\alpha - a(t^2 - \eta^2)}{\sqrt{1 - [\alpha - a(t^2 - \eta^2)]^2}} dt$$

we have $y'(x_0)=0$, $x_0 < 0$. For $y \rightarrow (\eta^2 + \alpha/a)^{1/2} = y_M$, it follows $x \rightarrow x_M$, where

$$(8) \quad x_M = \int_{\eta}^{y_M} \frac{\alpha - a(t^2 - \eta^2)}{\sqrt{1 - [\alpha - a(t^2 - \eta^2)]^2}} dt > 0.$$

Making use of the Remark 4, we conclude that the solutions satisfying the initial conditions (i) is defined on the maximal interval (x_m, x_M) , where $x_m = 2x_0 - x_M$; we have $y'(x) \rightarrow \pm\infty$ for $x \rightarrow x_M$ or $x \rightarrow x_m$. The graph of this solution is of the form given in Fig. 1. Starting from the constructed solution, we obtain another solutions of (2) by periodicity (see the Remark 5).

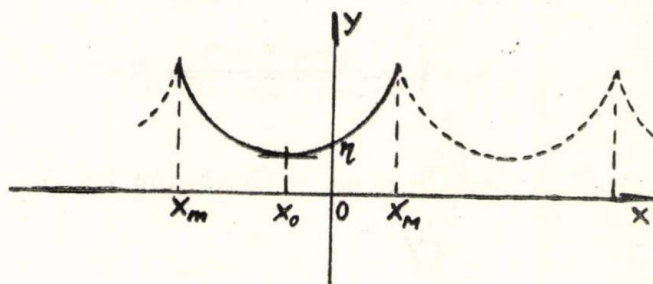


Fig. 1

(ii) $y(0) = \eta \geq 1/\sqrt{a}$, $y'(0) < 0$. This case is analogous with the preceding; to obtain the corresponding solution of (2), it must take the sign — in the second formula (6); the interval of variation for y is the same as in the preceding case; y_0 is again given by (7), but x_0 is the opposite of the integral in the second formula (7). The solution will be defined on the maximal interval (x_m, x_M) , with $x_M = 2x_0 - x_m$ and x_m given by the opposite of the integral in (8). The graph of this solution is given in Fig. 2.

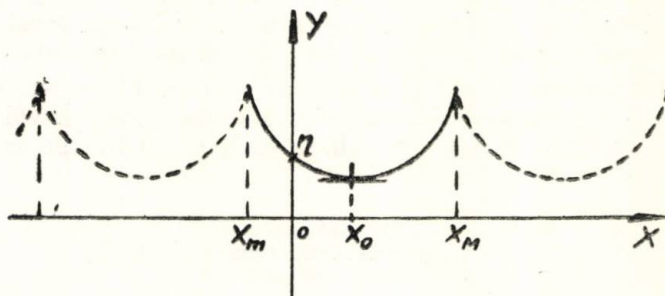


Fig. 2

(iii) $y(0) = \eta \in (0, 1/\sqrt{a})$, $y'(0) > 0$, $\alpha > 1 - a\eta^2$. This case is very similar to (i); the graph of the solution is sketched in Fig. 1.

(iv) $y(0) = \eta \in (0, 1/\sqrt{a})$, $y'(0) < 0$, $\alpha > 1 - a\eta^2$. This case is completely analogous to (ii); the graph of the solution is sketched in the Fig. 2.

(v) $y(0) = \eta \in (0, 1/\sqrt{a})$, $y'(0) < 0$, $\alpha = 1 - a\eta^2$. This time the interval of variation for y is $(0, (\eta^2 + \alpha/a)^{1/2}]$ and the corresponding integral curve has the equation

$$(9) \quad x = \int_y^{\eta} \frac{\alpha - a(t^2 - \eta^2)}{\sqrt{1 - [\alpha - a(t^2 - \eta^2)]^2}} dt.$$

The expression under the radical may be written as $t^2(b^2 - a^2t^2)$, where $b^2 = 2a\eta^2 + 2a\alpha$, and consequently for $y \rightarrow 0$ the integral (9) tends to $+\infty$. We obtain a solution of (2) defined on $(x_m, +\infty)$ where

$$(10) \quad x_m = \int_{\sqrt{\eta^2 + \frac{\alpha}{a}}}^{\eta} \frac{\alpha - a(t^2 - \eta^2)}{\sqrt{1 - [\alpha - a(t^2 - \eta^2)]^2}} dt < 0$$

and $y'(x) \rightarrow -\infty$ for $x \rightarrow x_m$. The graph is sketched in Fig. 3.

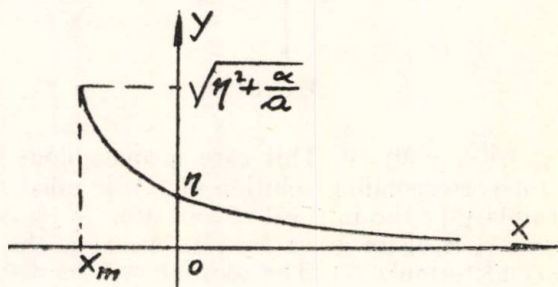


Fig. 3

(vi) $y(0) = \eta \in (0, 1/\sqrt{a})$, $y'(0) > 0$, $\alpha = 1 - a\eta^2$. This case is very similar to the preceding. The solution will be defined on the interval $(-\infty, x_M)$, with $x_M > 0$ given by the opposite of the integral from formula (10).

(vii) $y(0) = \eta \in (0, 1/\sqrt{a})$, $y'(0) < 0$, $\alpha < 1 - a\eta^2$. The interval of variation for y is $(-(\eta^2 + \alpha/a)^{1/2}, (\eta^2 + \alpha/a)^{1/2})$ and the solution will be defined on the maximal interval (x_m, x_M) , where x_m is given by the same formula (10) and x_M is given by

$$(11) \quad x_M = \int_{-\sqrt{\eta^2 + \frac{\alpha}{a}}}^{\eta} \frac{\alpha - a(t^2 - \eta^2)}{\sqrt{1 - [\alpha - a(t^2 - \eta^2)]^2}} dt > 0.$$

Obviously, $y'(x) \rightarrow -\infty$ for $x \rightarrow x_m$ and $y'(x) \rightarrow -\infty$ for $x \rightarrow x_M$. The graph will have an inflexion point in $(\xi, 0)$, where

$$(12) \quad \xi = \int_0^{\eta} \frac{\alpha - a(t^2 - \eta^2)}{\sqrt{1 - [\alpha - a(t^2 - \eta^2)]^2}} dt > 0$$

and is represented in the Fig. 4. Of course, making use of the Remark 5 and starting from the above solution, we can construct by periodicity ano-

ther solution of (2), defined on the intervals $(x_m + kT, x_M + kT)$, where $k = \text{integer}$ and $T = x_M - x_m$.

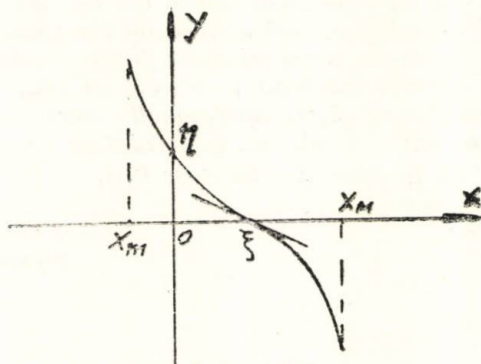


Fig. 4

(viii) $y(0) = \eta \in (0, 1/\sqrt{a})$, $y'(0) > 0$, $\alpha < 1 - a\eta^2$. This case is similar with the preceding and we shall omit to discuss it here.

(ix) $y(0) = \eta > 0$, $y'(0) = 0$. This is the case when $\alpha = 1$; the interval of variation for y is $[\eta, (\eta^2 + 1/a)^{1/2}]$. The solution will be increasing on the interval $(0, x_M)$, where

$$(13) \quad x_M = \int_{\eta}^{y_M} \frac{1 - a(t^2 - \eta^2)}{\sqrt{1 - [1 - a(t^2 - \eta^2)]^2}} dt, \quad y_M = \sqrt{\eta^2 + \frac{1}{a}},$$

and his domain of definition is $(-x_M, x_M)$, if we take into account the Remark 4. For $x \rightarrow x_M$ it follows $y(x) \rightarrow y_M$ and $y'(x) \rightarrow +\infty$. We shall not sketch the graph of the corresponding solution, but we remark that it is as in the Fig. 1 (with $x_0 = 0$). After some elementary computation, we obtain for the maximal abscissa x_M the formula

$$(14) \quad x_M = \varphi(k)\eta, \quad \varphi(k) = \int_1^{\sqrt{1+1/k}} \frac{1 - k(s^2 - 1)}{\sqrt{2k(s^2 - 1) - k^2(s^2 - 1)^2}} ds,$$

where $k = a\eta^2$. Of course, it follows

$$(15) \quad x_M > \eta \int_1^{\sqrt{1+1/k}} \frac{1 - k(s^2 - 1)}{\sqrt{2k(s^2 - 1)}} ds.$$

(x) $y(0) = 0$, $y'(0) \neq 0$. Nothing is new in this case, which is very similar to the cases (vii)–(viii) and we shall omit to repeat the same considerations which were made there.

Finally, let us remark that the situation when $y(0) < 0$ is not necessary to be studied (see the Remark 1). We conclude this Note, stating the following

- Proposition.** a) Every solution of the Eq. (2) is bounded.
 b) There exist no solutions $\neq 0$ defined on the whole real axis.
 c) Most of the solutions are defined on finite intervals, but there exist solutions defined on infinite intervals $(-\infty, x_M)$ or $(x_m, +\infty)$.
 d) The null solution of (2) is conditionally stable, i.e. for $y(0) = \text{small}$ and $y'(0)$ suitable choosen, the solution is defined on $[0, \infty)$ and $y(x)$, $y'(x)$ remain small (and tend to zero) (see the case (v)).

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DESPRE ECUAȚIA $f(y')y'' = 2ay$

(Rezumat)

Se enumeră cîteva proprietăți generale ale acestei ecuații, în cazul cînd $f=f(u)$ este o funcție pară. Se tratează pe larg cazul cînd $f(u) = (1+u^2)^{-3/2}$, adică se găsesc curbele plane pentru care curbura este proporțională cu ordonata y .

A COMPENSATED COMPACTNESS RESULT

BY

I. I. VRABIE

1. Introduction. A simple argument involving Mazur's Theorem shows that for a linear operator A acting between two real Banach spaces the following conditions are equivalent:

- (i) the graph of A is weakly $\times$ weakly closed,
- (ii) the graph of A is strongly $\times$ strongly closed.

In the nonlinear case, this very simple but extremely useful equivalence is no longer true since the graph of a nonlinear operator is not convex. Therefore, a major difficulty we have to overcome in order to prove existence results for nonlinear PDEs by using compactness arguments rests in the fact that we must check directly the condition (i) above. In many circumstances, this can be done with the help of some specific a priori estimates combined with the so-called compensated compactness initiated by Murat [4]. For details, further developments and applications in this respect, see also [1], [5] and the references therein.

Our main result, which has been suggested by a problem arisen in the study of the heat conduction in materials with memory, is a useful tool in checking the condition (i) for a broad class of nonlinear partial differential operators. See [6, Theorem 5.4.1, p. 289]. Other interesting applications will appear elsewhere.

2. The Main Result. Now we can proceed to the statement of our main result.

Theorem. Let (Ω, Σ, μ) be a σ -finite measure space, let (f_n) be a sequence of functions from Ω into $\overline{\mathbb{R}}$, and let (g_n) be a sequence in $L^1(\Omega, \mu; \mathbb{R})$. If $\lim f_n = f$ μ -a.e. in Ω and both $\lim g_n = g$ and $\lim f_n g_n = h$ weakly in $L^1(\Omega, \mu; \mathbb{R})$, then $h = f \cdot g$ μ -a.e. in Ω .

Remark. It is interesting to note in the theorem above the f_n , s and f are neither in $L^1(\Omega, \mu; \mathbb{R})$, nor even μ -measurable from Ω into $\mathbb{R}$. In the specific case when Ω is a compact topological space, μ is a Radon measure on Ω and (f_n) is a sequence of functions from Ω in $\overline{\mathbb{R}}$ which, in addition, are μ -measurable, the conclusion of our main result follows rather easily from a successive application of Egoroff's Theorem and of Luzin's Theorem. See [6, Lemma 5.4.1, 290].

3. Proof. Since (Ω, Σ, μ) is σ -finite, we have $\Omega = \bigcup_{k \in \mathbb{N}} \Omega_k$ with $\Omega_k \in \Sigma$ and $\mu(\Omega_k) < +\infty$ for each $k \in \mathbb{N}$, and therefore it suffices to show that,

for each $k \in \mathbb{N}$, $h = f.g$ μ -a.e. in Ω_k . Consequently, we may assume without loss of generality that (Ω, Σ, μ) itself is a finite measure space. Obviously, $\{g_n; n \in \mathbb{N}\}$ is weakly relatively sequentially compact in $\mathcal{L}^1(\Omega, \mu; \mathbb{R})$. From Dunford-Pettis' Theorem [2, Theorem 1, p. 101], it then follows that $\{|g_n|; n \in \mathbb{N}\}$ is weakly relatively sequentially compact in $\mathcal{L}^1(\Omega, \mu; \mathbb{R})$. Hence there exist $g \in \mathcal{L}^1(\Omega, \mu; \mathbb{R})$ and a subsequence of $(|g_n|)$ — denoted for simplicity again by $(|g_n|)$ — such that

$$\lim |g_n| = g^* \text{ weakly in } \mathcal{L}^1(\Omega, \mu; \mathbb{R}).$$

Next, from Mazur's Theorem [3, Corollary, p. 36], we conclude that there exists a sequence $(\tilde{g}_n)$ of convex combinations of $\{|g_n|, |g_{n+1}|, \dots\}$ such that

$$\lim \tilde{g}_n = g^* \text{ strongly in } \mathcal{L}^1(\Omega, \mu; \mathbb{R}).$$

To fix the ideas set

$$\tilde{g}_n = \sum_{k=n}^{\infty} \lambda_{n,k} |g_k|,$$

where $\lambda_{n,k} \in [0, 1]$, $\lambda_{n,k} \neq 0$ only for a finite number of indices k , and $\sum_{k=n}^{\infty} \lambda_{n,k} = 1$ for each $n \in \mathbb{N}$.

Since each strongly convergent sequence in $\mathcal{L}^1(\Omega, \mu; \mathbb{R})$ has at least one subsequence which converges μ -a.e. in Ω to the same limit, we may assume without loss of generality that

$$\lim \tilde{g}_n = g^* \mu\text{-a.e. in } \Omega.$$

Obviously

$$\lim \sum_{k=n}^{\infty} \lambda_{n,k} g_k = g \text{ weakly in } \mathcal{L}^1(\Omega, \mu; \mathbb{R}),$$

and therefore, again by Mazur's Theorem, there exists a sequence (G_n) of convex combinations of $\{\sum_{k=n}^{\infty} \lambda_{n,k} g_k, \sum_{k=n+1}^{\infty} \lambda_{n+1,k} g_k, \dots\}$ such that

$$\lim G_n = g \text{ strongly in } \mathcal{L}^1(\Omega, \mu; \mathbb{R}).$$

Clearly, (G_n) is given by

$$G_n = \sum_{k=n}^{\infty} \xi_{n,k} \sum_{j=k}^{\infty} \lambda_{j,j} g_j = \sum_{p=n}^{\infty} \eta_{n,p} g_p,$$

where $\xi_{n,k}, \eta_{n,p} \in [0, 1]$, $\xi_{n,k} \neq 0$ only for a finite number of indices k , $\eta_{n,p} \neq 0$ only for a finite number of indices p , and $\sum_{k=n}^{\infty} \xi_{n,k} = \sum_{p=n}^{\infty} \eta_{n,p} = 1$, for each $n \in \mathbb{N}$.

Here we may assume without loss of generality (by extracting a subsequence if necessary) that

$$(1) \quad \lim G_n = g \mu\text{-a.e. in } \Omega.$$

Let us denote by

$$\tilde{G}_n = \sum_{p=n}^{\infty} \eta_{n,p} |g_p|, \quad F_n = \sum_{p=n}^{\infty} \eta_{n,p} f_p, \quad H_n = \sum_{p=n}^{\infty} \eta_{n,p} f_p g_p,$$

for each $n \in \mathbb{N}$. We then have

$$(2) \quad \lim \tilde{G}_n = g^* \quad \mu\text{-a.e.} \quad \text{in } \Omega,$$

$$(3) \quad \lim F_n = f \quad \mu\text{-a.e.} \quad \text{in } \Omega,$$

and

$$(4) \quad \lim H_n = h \quad \text{weakly in } \mathcal{L}^1(\Omega, \mu; \mathbb{R}).$$

Now, let us observe that

$$\begin{aligned} |H_n(x) - F_n(x)G_n(x)| &= \left| \sum_{p=n}^{\infty} \eta_{n,p} g_p(x) \sum_{q=n}^{\infty} \eta_{n,q} [f_p(x) - f_q(x)] \right| \leq \\ &\leq \sum_{p=n}^{\infty} \eta_{n,p} |g_p(x)| \sum_{q=n}^{\infty} \eta_{n,q} |f_p(x) - f_q(x)|, \end{aligned}$$

for each $n \in \mathbb{N}$ and $x \in \Omega$. Hence

$$|H_n(x) - F_n(x)G_n(x)| \leq \sup\{|f_p(x) - f_q(x)|; p \geq n, q \geq n\} \tilde{G}_n(x)$$

for each $n \in \mathbb{N}$ and $x \in \Omega$. Taking into account of (1), (2) and (3), we easily conclude that

$$\lim H_n = f \cdot g \quad \mu\text{-a.e.} \quad \text{in } \Omega.$$

Finally, inasmuch as $\lim H_n = h$ weakly in $\mathcal{L}^1(\Omega, \mu; \mathbb{R})$, the last relation in conjunction with Dunford-Pettis' Theorem [2, Theorem 1, p. 101] shows that (H_n) satisfies the hypotheses of Vitali's Theorem [7, Theorem VIII.4.4, p. 210]. Consequently we have

$$\lim H_n = f \cdot g \quad \text{strongly in } \mathcal{L}^1(\Omega, \mu; \mathbb{R}),$$

relation which along with (4) completes the proof.

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UN REZULTAT DE COMPACTITATE COMPENSATĂ

(Rezumat)

Fie (Ω, Σ, μ) un spațiu cu măsură σ -finită, (f_n) un șir de funcții de la Ω în $\overline{\mathbb{R}}$, și (g_n) un șir din $\mathcal{L}^1(\Omega, \mu; \mathbb{R})$. Dacă $\lim f_n = f$ μ -a.p.t. pe Ω , și $\lim g_n = g$, $\lim f_n g_n = h$ slab în $\mathcal{L}^1(\Omega, \mu; \mathbb{R})$, atunci $h = f \cdot g$ μ -a.p.t. pe Ω .

EXTENSIONS D'UN THÉORÈME DE S. K. SAMANTA POUR DES SUITES D'APPLICATIONS MULTIVOQUES

PAR

NICOLETA NEGOESCU

1. Quelques théorèmes de point fixe pour des applications multivoques contractives dans des espaces métriques et dans des espaces de Banach sont démontrés par L.S. Dube [1], L.S. Dube et S.P. Singh [2], S. Reich [5], B.K. Ray [4], K. Yanagi [8].

En 1981 S.K. Samanta [6] a démontré pour la première fois un théorème de point fixe pour une application multivoque $T: C \rightarrow \text{Cpt}(X)$, où C est un sousensemble faiblement compact d'un espace de Banach qui satisfait à la condition de Opial.

Dans cette Note nous avons démontré deux théorèmes de point fixe commun pour des suites d'applications multivoques définies sur un sousensemble faiblement compact d'un espace de Banach X qui satisfait à la condition de Opial:

a) avec des valeurs dans l'ensemble $\text{Cpt}(X)$ des sousensembles compacts, nonvides de X (qui étend le résultat de Samanta [6]) et

b) avec des valeurs dans l'ensemble $CB(X)$ des sousensembles nonvides, bornés et fermés de X .

2. Nous rappelons quelques notions qui nous sont nécessaires dans les démonstrations de ces résultats.

Définition 1. Un espace de Banach X satisfait à la condition de Opial [3] si pour toute suite $(x_n) \subset X$ qui est convergente faiblement à x il suit l'inégalité

$$\lim_n \inf \|x_n - x\| < \lim_n \inf \|x_n - y\| \quad \text{pour tout } y \neq x.$$

Définition 2. $H(A, B) = \max \{ \sup_{x \in A} D(x, B), \sup_{y \in B} D(y, A) \}$ pour $\forall A, B \in CB(X)$ ou $\forall A, B \in \text{Cpt}(X)$, où $D(x, B) = \inf_{y \in B} \|x - y\|$, s'appelle la distance de Hausdorff-Pompeiu entre les ensembles A, B de $\text{Cpt}(X)$ ou de $CB(X)$, induite par la métrique $d(x, y) = \|x - y\|$, $\forall x, y \in X$.

Lemme 1. a) Soient $A, B \in \text{Cpt}(X)$; alors pour tout $a \in A$ il existe $b \in B$ tel que $\|a - b\| \leq H(A, B)$.

1. b) Soient $A, B \in CB(X)$ et $k \in (1, +\infty)$ donné. Alors pour tout $a \in A$, il existe $b \in B$ tel que $\|a - b\| \leq kH(A, B)$.

3. Nous allons démontrer deux théorèmes de point fixe commun dans les conditions énoncées dans § 1.

Théorème 1. Soient X un espace de Banach qui satisfait à la condition de Opial, $C \subset X$ un sousensemble faiblement compact et $T_n: C \rightarrow \text{Cpt}(X)$ une suite d'applications multivoques qui satisfont aux conditions suivantes :

(i) $\sup_{n \in \mathbb{N}^*} \inf_{x \in C} D(x, T_n x) = 0$ (où $D(x, T_n x) = \inf_{y \in T_n x} \|x - y\|$, $T_n x \in \text{Cpt}(X)$),
v. Def. 2);

(ii) ils existent les nombres $a, b, c \geq 0$ avec $a + 2b + 2c \leq 1$ tels que
 $H(T_i x, T_j y) \leq a \|x - y\| + b (D(x, T_i x) + D(y, T_j y)) + c (D(x, T_j y) + D(y, T_i x))$,
 $\forall x, y \in C$ et $\forall i, j \in \mathbb{N}^*$.

Alors les applications multivoques (T_n) ont un point fixe commun $u \in C$.
La démonstration de ce théorème a trois étapes.

I. Soit T_{n+1} l'une des applications multivoques de l'énoncé du Théorème 1. La condition (i) implique que $\inf_{x \in C} D(x, T_{n+1} x) = 0$. Soit $\forall x_0 \in C$ tel que $D(x_0, T_{n+1} x_0) > 0$ et soit $x_1 \in C$ tel que $D(x_0, T_{n+1} x_0) > D(x_1, T_{n+1} x_1) > 0$ et, en général, soit $x_m \in C$ tel que $D(x_{m-1}, T_{n+1} x_{m-1}) > D(x_m, T_{n+1} x_m)$. Les éléments $x_m \in C$, par la suite de l'hypothèse (i), peuvent être choisis tels que $D(x_m, T_{n+1} x_m)$ est convergente à 0 ($\lim_{m \rightarrow \infty} D(x_m, T_{n+1} x_m) = 0$).

II. Car C est faiblement compact, il existe une sous-suite de (x_n) qui est convergente faiblement à $u \in C$. Pour ne pas compliquer l'écriture, nous notons aussi cette sous-suite par (x_n) .

Soient n donné ($n \in \mathbb{N}$) et l'ensemble $T_{n+1} x_m$ avec $m \in \mathbb{N}$. Car l'ensemble $T_{n+1} x_m$ est compact, soit $y_m \in T_{n+1} x_m$ tel que $D(x_m, T_{n+1} x_m) = \|x_m - y_m\|$, c'est à dire y_m est un élément de la meilleure approximation de x_m dans $T_{n+1} x_m$ (la compacité de l'ensemble $T_{n+1} x_m$ implique le fait que cet ensemble est borné et fermé et y_m est un élément de la meilleure approximation, c'est à dire $T_{n+1} x_m$ sont des ensembles proximinales).

(1) Pour $y_m \in T_{n+1} x_m$ il existe $z_m \in T_{n+1} u$ (v. Lemme 1a) qui a la propriété $\|y_m - z_m\| \leq H(T_{n+1} x_m, T_{n+1} u)$. De (ii) il suit

$$(2) \quad H(T_{n+1} x_m, T_{n+1} u) \leq a \|x_m - u\| + b (D(x_m, T_{n+1} x_m) + D(u, T_{n+1} u)) + c (D(x_m, T_{n+1} u) + D(u, T_{n+1} x_m)),$$

c'est à dire

$$\|y_m - z_m\| \leq a \|x_m - u\| + b (D(x_m, T_{n+1} x_m) + D(u, T_{n+1} u)) + c (D(x_m, T_{n+1} u) + D(u, T_{n+1} x_m)),$$

ou, en employant les éléments de la meilleure approximation de x_m de $T_{n+1} x_m$, nous obtenons

$$\|y_m - z_m\| \leq a \|x_m - u\| + b (\|x_m - y_m\| + \|u - z_m\|) + c (\|x_m - z_m\| + \|u - y_m\|).$$

En employant l'inégalité du triangle entre les parenthèses, il suit

$$(3) \quad \|y_m - z_m\| \leq a \|x_m - u\| + b (\|x_m - y_m\| + \|u - x_m\| + \|x_m - z_m\|) + c (\|x_m - z_m\| + \|u - x_m\| + \|x_m - y_m\|),$$

ou

$$||x_m - z_m|| \leq ||x_m - y_m|| + ||y_m - z_m|| \leq ||x_m - y_m|| + a||x_m - u|| + b(||x_m - y_m|| + ||u - x_m|| + ||x_m - z_m||) + c(||x_m - z_m|| + ||u - x_m|| + ||x_m||),$$

c'est à dire

$$(1 - b - c)||x_m - z_m|| \leq (a + b + c)||x_m - u|| + (1 + b + c)||x_m - y_m||.$$

En divisant par $1 - b - c > 0$ (le cas $a = b = c = 0$ est exclus car alors il suivrait que $y_m \equiv z_m$, c'est à dire $T_{n+1} x_m \cap T_{n+1} u \neq \emptyset$ ce que n'est pas toujours vrai pour $\forall m$) nous obtenons

$$(4) \quad ||x_m - z_m|| \leq \frac{a + b + c}{1 - b - c} ||x_m - u|| + \frac{1 + b + c}{1 - b - c} ||x_m - y_m||.$$

Soit w un élément arbitraire de X . Alors $||x_m - w|| \leq ||x_m - z_m|| + ||z_m - w|| \leq ||z_m - w|| + \frac{a + b + c}{1 - b - c} ||x_m - u|| + \frac{1 + b + c}{1 - b - c} ||x_m - y_m||$, c'est à dire il suit

$$(5) \quad ||x_m - w|| \leq \frac{a + b + c}{1 - b - c} ||x_m - u|| + \frac{1 + b + c}{1 - b - c} ||x_m - y_m|| + ||z_m - w||.$$

III. Car $\lim_{m \rightarrow \infty} D(x_m, T_{n+1} x_m) = 0$, il suit que $\lim_{m \rightarrow \infty} ||x_m - y_m|| = 0$. De la compacité de l'ensemble $T_{n+1} u$ et de (1), $z_m \in T_{n+1} u$ et il existe une sous-suite (z_{m_p}) de (z_m) , $z_{m_p} \rightarrow v$, donc

$$(6) \quad v \in T_{n+1} u.$$

Alors de l'inégalité (5) avec $w = v$, où $z_{m_p} \rightarrow v$, il suit

$$\liminf_{p \rightarrow \infty} ||x_{m_p} - v|| \leq \liminf_{p \rightarrow \infty} \left(\frac{a + b + c}{1 - b - c} ||x_{m_p} - u|| + \frac{1 + b + c}{1 - b - c} ||x_{m_p} - y_{m_p}|| + ||z_{m_p} - v|| \right).$$

Car la suite numérique $(||x_{m_p} - y_{m_p}||)$ est convergente à 0 et $||z_{m_p} - v|| \rightarrow 0$ conformément aux observations antérieures et au fait que $\liminf_{n \rightarrow \infty} (a_n + b_n + c_n) = \liminf_{n \rightarrow \infty} a_n + \liminf_{n \rightarrow \infty} b_n + \liminf_{n \rightarrow \infty} c_n$ quand (b_n) et (c_n) sont des suites numériques convergentes et (a_n) est une suite numérique bornée, il suit que

$$\liminf_{p \rightarrow \infty} ||x_{m_p} - v|| \leq \frac{a + b + c}{1 - b - c} \liminf_{p \rightarrow \infty} ||x_{m_p} - u||.$$

Donc on obtient

$$(7) \quad \liminf_{p \rightarrow \infty} ||x_{m_p} - u|| \geq \liminf_{p \rightarrow \infty} ||x_{m_p} - v||$$

car $a + 2b + 2c \leq 1$ est équivalente à $(a + b + c)/(1 - b - c) \leq 1$.

Mais l'espace de Banach X satisfait à la condition Opial et il suit que $v=u$. Alors de (6) on obtient que $u \in T_{n+1}u$, c'est à dire u est un point fixe pour T_{n+1} pour $\forall n \in \mathbb{N}$. Donc $u \in C$, la limite faible de la suite (x_{m_p}) , est un point fixe commun pour toutes les applications multivoques T_n , $n \in \mathbb{N}^*$.

Un théorème analogue du Théorème 1 est vrai pour une suite d'applications multivoques $T_n: C \rightarrow CB(X)$ ($CB(X) \supset Cpt(X)$), mais à cause du Lemme 1 b) le calcul présente certaines modifications. Le même Lemme 1 impose la restriction de la condition (ii) du Théorème 1 à $a+2b+2c < 1$.

Théorème 2. Soient C un sousensemble faiblement compact d'un espace de Banach X qui satisfait à la condition de Opial et $T_n: C \rightarrow CB(X)$ une suite d'applications multivoques qui ont les propriétés

$$(i) \quad \sup_{n \in \mathbb{N}^*} (\inf D(x, T_{n+1}x)) = 0,$$

(ii) ils existent les nombres $a, b, c \geq 0$, $a+2b+2c < 1$ tels que $H(T_i x, T_j y) \leq a||y-x|| + b(D(x, T_i x) + D(y, T_j y)) + c(D(x, T_j y) + D(y, T_i x))$ pour $\forall x, y \in C$.

Alors les applications multivoques T_n ont un point fixe commun $u \in C$.

La démonstration de ce théorème a aussi trois étapes.

Dans la première étape on procède de la même manière que dans l'étape I du Théorème 1.

Dans l'étape II on choisit la suite (y_m) , $y_m \in T_{n+1}x_m$ avec la propriété $D(x_m, T_{n+1}x_m) = ||x_m - y_m||$, c'est à dire y_m est un élément de la meilleure approximation de x_m dans $T_{n+1}x_m$ et $z_m \in T_{n+1}u$.

Du Lemme 1 b) et de (ii) il suit

$$(2') \quad ||y_m - z_m|| \leq kH(T_{n+1}x_m, T_{n+1}u) \leq ka||x_m - u|| + kb(D(x_m, T_{n+1}x_m) + D(u, T_{n+1}u)) + kc(D(x_m, T_{n+1}u) + D(u, T_{n+1}x_m)),$$

où $k \in (1, \infty)$ fixé tel que $k(a+2b+2c) \leq 1$, c'est à dire $a+2b+2c < 1$.

III. Soit v la limite (forte) d'une sous-suite (z_{m_p}) de $(z_m) \subset T_{n+1}u$, (compact) nous avons

$$(6') \quad v \in T_{n+1}u$$

et

$$||x_{m_p} - v|| \leq \frac{k(a+b+c)}{1-k(b+c)} ||x_{m_p} - u|| + \frac{1+k(b+c)}{1-k(b+c)} ||x_{m_p} - y_{m_p}|| + ||z_{m_p} - v||.$$

L'inégalité $k(a+2b+2c) \leq 1$ étant équivalente à $k(a+b+c)/[1-k(b+c)] \leq 1$ il suit

$$\liminf_{p \rightarrow \infty} ||x_{m_p} - u|| \geq \liminf_{p \rightarrow \infty} ||x_{m_p} - v||.$$

Mais l'espace X satisfait à la condition Opial et alors $v=u$ et de (6') on a que u est un point fixe pour l'application multivoque T_{n+1} , donc u est un point fixe commun pour toutes les applications $T_n: C \rightarrow CB(X)$, $n \in \mathbb{N}^*$.

4. Observations. 1) Si (T_n) est une suite de fonctions, alors $T_n x$ (pour n fixé) est un point (donc un ensemble compact). Alors on obtient du Théorème 1 un résultat analogue pour une suite de fonctions.

2) Si dans le Théorème 1 et dans le résultat de l'observation 1) nous prenons $T_n = T$ alors nous obtenons, comme des cas particuliers, le Théorème 1.2 de S.K. Samanta [6] et un théorème analogue pour une seule fonction.

3) Si (T_n) est une suite de fonctions, alors $T_n x$ est un point (donc un ensemble borné et fermé) et donc du Théorème 2 on obtient un théorème analogue pour une suite de fonctions.

4) Si dans le Théorème 2 et dans le résultat de l'observation 3) nous prenons $T_n = T$, alors nous obtenons, comme des conséquences, des résultats analogues pour une seule application multivoque ou pour une seule fonction.

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EXTENSIUNI ALE UNEI TEOREME DE S. K. SAMANTA PENTRU ȘIRURI DE MULTIFUNCȚII

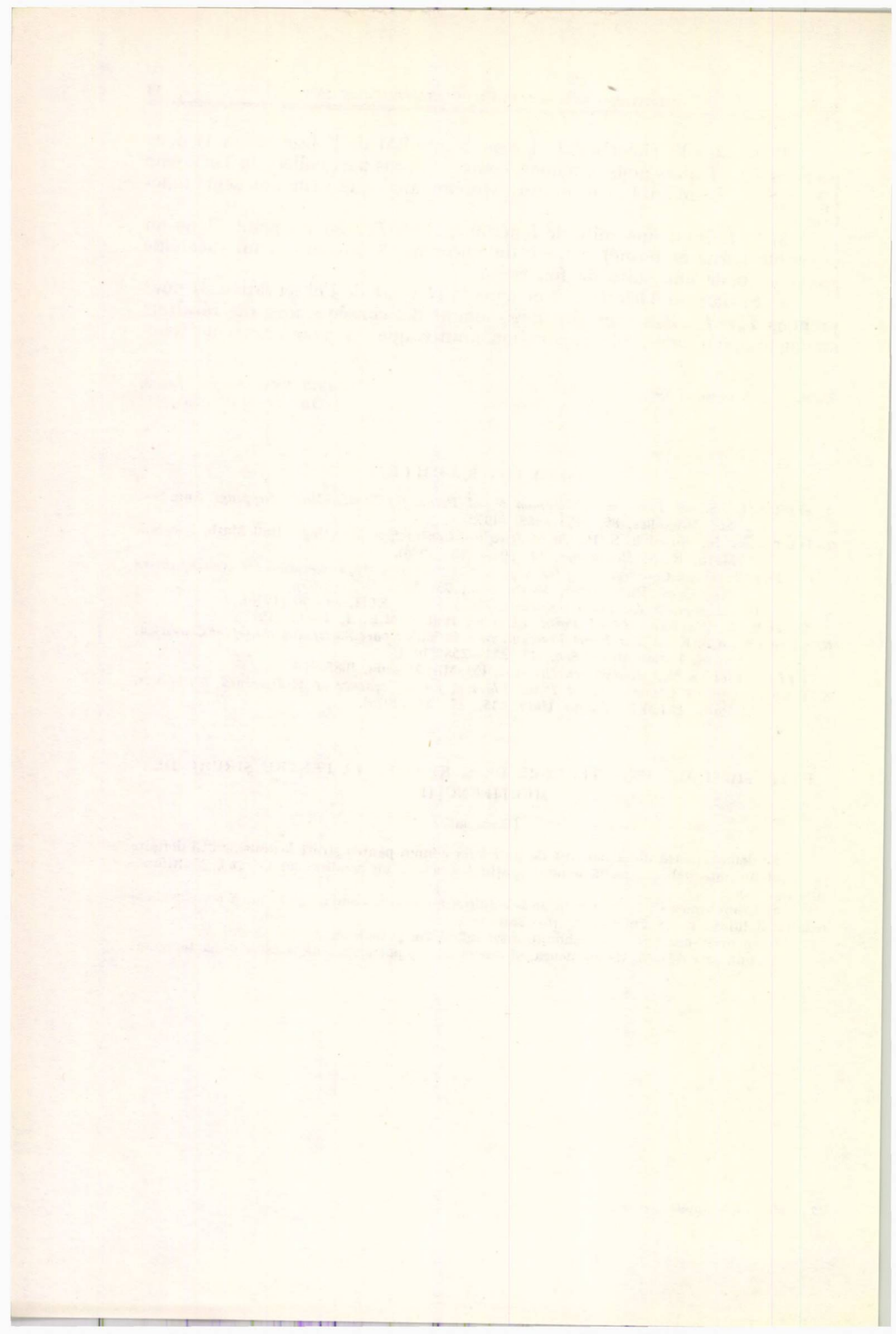
(Rezumat)

Se demonstrează două teoreme de punct fix comun pentru șiruri de multifuncții definite pe o submulțime slab compactă a unui spațiu Banach X cu condiția lui Opial. Multifuncțiile iau valori:

a) în mulțimea $Cpt(X)$ a tuturor submulțimilor nevide compacte ale lui X (care extinde rezultatul lui S. K. Samanta [6]) sau

b) în mulțimea $CB(X)$ a submulțimilor mărginite și închise, nevide ale lui X .

Se pun în evidență, de asemenea, și câteva cazuri particulare ale acestor două teoreme.



ON PERIODIC SOLUTIONS OF A VOLTERRA SYSTEM

BY

I. ONCIULESCU

1. Statement of the Problem. Let $y(t)$ and $z(t)$ be the densities of a prey and respectively a predator species living in a same environment. As one knows [5] the differential system describing the interdependence of the two species is

$$(1) \quad \begin{aligned} \frac{dy}{dt} &= y(a - bz), \\ \frac{dz}{dt} &= z(-c + dy), \end{aligned}$$

a, b, c, d being, in the above quoted book of Volterra and in many other papers dealing with this topics, given constants. As result of this system, the solution is a periodic one, for every pair of initial values y_0, z_0 of y and z .

Our aim is to find some systems having the same form as (1) but with a, b, c, d depending on time, the solutions of which remain periodic. To this end, we will investigate first the case when, for given a, b, c, d satisfying some restrictive conditions, there exist initial values y_0, z_0 of y and z and a time T , such that

$$(2) \quad y(T) = y_0, \quad z(T) = z_0.$$

After that some systems of the form (1) with periodic solution when a, b, c, d depend on time can easily be obtained.

2. Hypotheses and Results. We suppose:

(H₁) The given functions a, b, c, d in (1) are bounded and continuous on $\mathbb{R}_+$ and positive, i.e.,

$$\begin{aligned} 0 < \bar{a} \leq a(t) \leq \bar{\bar{a}}, & \quad 0 < \bar{b} \leq b(t) \leq \bar{\bar{b}}, \\ 0 < \bar{c} \leq c(t) \leq \bar{\bar{c}}, & \quad 0 < \bar{d} \leq d(t) \leq \bar{\bar{d}}; \end{aligned}$$

(H₂) The equation

$$\int_0^T a(\tau) d\tau = \log \frac{\bar{d} y_0}{c} / \left[\exp \left[y_0 \int_0^T d(\tau) \exp \left(\int_0^\tau a(s) ds \right) d\tau \right] \right],$$

where $T > 0$ is a constant, has, in the unknown y_0 , two solutions.

Remark. Such a constant T always exists (eventually sufficiently small).

With these hypotheses we will prove

Theorem 1. *If the hypotheses (H_1) and (H_2) are satisfied, then there exists a pair (y_0, z_0) of positive initial values of (y, z) such that*

$$(3) \quad y(T) = y_0 > 0, \quad z(T) = z_0 > 0$$

hold.

Theorem 2. *If the conditions (H_1) , (H_2) are satisfied and a, b, c, d are T -periodic functions, then the system (1) has T -periodic solutions.*

3. Proofs. Let y_0 and z_0 be two positive numbers. We denote $(y(t, y_0, z_0), z(t, y_0, z_0))$, $t > 0$, the solution of the system (1) with the property

$$y(0, y_0, z_0) = y_0, \quad z(0, y_0, z_0) = z_0.$$

Obviously

$$(1') \quad \begin{aligned} y(t, y_0, z_0) &= y_0 \exp \int_0^t (a(\tau) - b(\tau)z(\tau, y_0, z_0)) d\tau, \\ z(t, y_0, z_0) &= z_0 \exp \int_0^t (-c(\tau) + d(\tau)y(\tau, y_0, z_0)) d\tau. \end{aligned}$$

From (1') it results immediately

$$y(t, y_0, z_0) > 0, \quad z(t, y_0, z_0) > 0, \quad \text{for every } t \geq 0$$

and

$$\begin{aligned} y(t, y_0, z_0) &\leq y_0 A(t), & A(t) &= \exp \int_0^t a(\tau) d\tau, \\ z(t, y_0, z_0) &\geq z_0 C(t), & C(t) &= \exp \int_0^t (-c(\tau)) d\tau. \end{aligned}$$

Using these bounds of y and z again in (1') we obtain

$$(4) \quad \begin{aligned} y(t, y_0, z_0) &\leq y_0 A_1(t), & A_1(t) &= \exp \int_0^t (a(\tau) - b(\tau)z_0 C(\tau)) d\tau, \\ z(t, y_0, z_0) &\leq z_0 C_1(t), & C_1(t) &= \exp \int_0^t (-c(\tau) + d(\tau)y_0 A(\tau)) d\tau, \end{aligned}$$

and

$$(5) \quad \begin{aligned} y(t, y_0, z_0) &\geq y_0 A_2(t), & A_2(t) &= \exp \int_0^t (a(\tau) - b(\tau) z_0 C_1(\tau)) d\tau, \\ z(t, y_0, z_0) &\geq z_0 C_2(t), & C_2(t) &= \exp \int_0^t (-c(\tau) + d(\tau) y_0 A_2(\tau)) d\tau, \end{aligned}$$

and also

$$(6) \quad y(t, y_0, z_0) \leq y_0 A_3(t), \quad A_3(t) = \exp \int_0^t (a(\tau) - b(\tau) z_0 C_2(\tau)) d\tau.$$

We denote

$$D_1(t) = \{(y_0, z_0) ; y_0 > 0, z_0 > 0, A_1(t) < 1\},$$

$$D_2(t) = \{(y_0, z_0) ; y_0 > 0, z_0 > 0, A_2(t) > 1\},$$

$$D_3(t) = \{(y_0, z_0) ; y_0 > 0, z_0 > 0, C_1(t) < 1\},$$

$$D_4(t) = \{(y_0, z_0) ; y_0 > 0, z_0 > 0, C_2(t) > 1\}.$$

It results, from (4) and (5) that

$$\text{if } (y_0, z_0) \in D_1(t) \quad \text{then} \quad y(t, y_0, z_0) < y_0,$$

$$\text{if } (y_0, z_0) \in D_2(t) \quad \text{then} \quad y(t, y_0, z_0) > y_0,$$

$$\text{if } (y_0, z_0) \in D_3(t) \quad \text{then} \quad z(t, y_0, z_0) < z_0,$$

$$\text{if } (y_0, z_0) \in D_4(t) \quad \text{then} \quad z(t, y_0, z_0) > z_0.$$

We have

$$(7) \quad D_4(t) = \left\{ (y_0, z_0), y_0, z_0 > 0, \int_0^t (-c(\tau) + d(\tau) y_0 A_2(\tau)) d\tau > 0 \right\}.$$

Taking into account (H_1) , (4) and (5) it follows that

$$(8) \quad \int_0^t (-c(\tau) + d(\tau) y_0 A_2(\tau)) d\tau \geq -\bar{c}t + \bar{d}y_0t \left(\exp \left(-z_0 \int_0^t C_1(s) b(s) ds \right) \right).$$

If we denote

$$D'_4(t) = \left\{ (y_0, z_0), y_0, z_0 > 0, z_0 < \frac{\log \bar{d} + \log y_0 - \log \bar{c}}{\int_0^t b(s) C_1(s) ds} \right\},$$

then, from (7) and (8) we have $D_4' \subset D_4$. From (4) it also results

$$(9) \quad \frac{\log \bar{d}y_0 - \log \bar{c}}{\int_0^t b(s)C_1(s)ds} = \frac{\log \bar{d}y_0 - \log \bar{c}}{\left(\int_0^t b(s)C(s)ds\right) \exp \int_0^{\eta} d(\tau)y_0 A(\tau)d\tau} \geq \\ \geq \frac{\log \bar{d}y_0 - \log \bar{c}}{\left(\int_0^t b(s)C(s)ds\right) \left(\exp \left(\int_0^t d(\tau)A(\tau)d\tau\right)\right)},$$

where $\eta \in [0, t]$ is obtained by a mean value theorem. Then, denoting

$$D_4''(t) = \left\{ (y_0, z_0), y_0, z_0 > 0, z_0 < \frac{\log \bar{d}y_0 - \log \bar{c}}{\left(\int_0^t b(s)C(s)ds\right) \left(\exp y_0 \int_0^t d(\tau)A(\tau)d\tau\right)} \right\},$$

from (9) it results $D_4'' \subset D_4'$ and hence $D_4'' \subset D_4$. The domains D_1, D_2, D_3, D_4'' have the forms as in Fig. 1. If the condition (H_2) is satisfied, then $D_1(T) \cap D_4''(T)$ has the form as in Fig. 1.

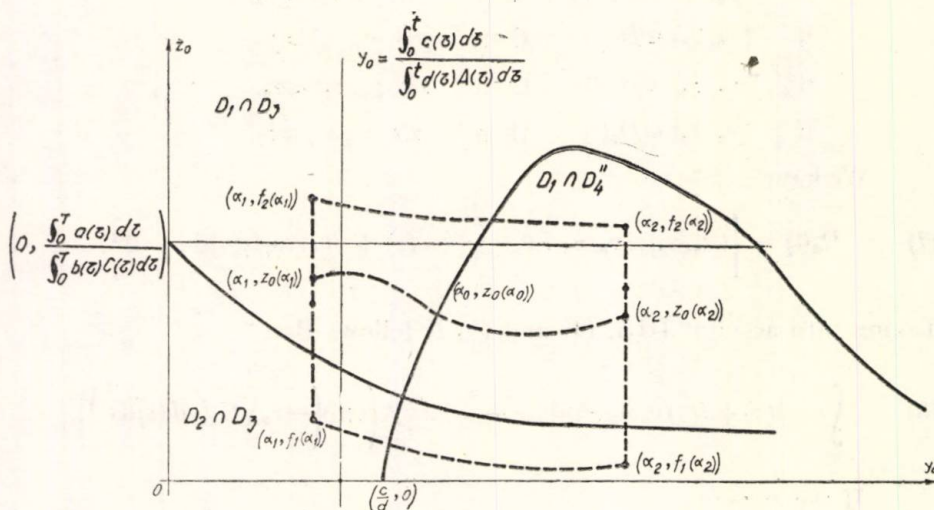


Fig. 1

We shall prove now the existence of a pair (y_0, z_0) so that $y(T, y_0, z_0) = y_0$ and $z(T, y_0, z_0) = z_0$. Taking into account the form of the domains $D_1, D_2, D_3, D_4'', D_1 \cap D_4''$, we can find $\alpha_1, \alpha_2 > 0$ and the continuous functions $f_1(\alpha)$ and $f_2(\alpha)$ in order to have

the segment $[(\alpha_1, f_1(\alpha_1)), (\alpha_1, f_2(\alpha_1))] \subset D_3(T)$,

the segment $[(\alpha_2, f_1(\alpha_2)), (\alpha_2, f_2(\alpha_2))] \subset D_4''$,

the curve $\{(y_0, z_0), y_0 = \alpha, z_0 = f_1(\alpha), \alpha \in [\alpha_1, \alpha_2]\} \subset D_2$,

and the curve $\{(y_0, z_0), y_0 = \alpha, z_0 = f_2(\alpha), \alpha \in [\alpha_1, \alpha_2]\} \subset D_1$.

Then

$$y(T, \alpha, f_1(\alpha)) - \alpha < 0 \quad \text{and} \quad y(T, \alpha, f_2(\alpha)) - \alpha > 0 \quad \text{for every } \alpha \in [\alpha_1, \alpha_2].$$

It results, as a consequence of the continuity of $y(T, y_0, z_0)$ as function on y_0, z_0 , that on the segment $[(\alpha, f_1(\alpha)), (\alpha, f_2(\alpha))]$ there exist some points $(\alpha, z_0(\alpha))$ having the properties

$$(A_1) \quad y(T, \alpha, z_0(\alpha)) - \alpha = 0,$$

and

(A_2) for every $\varepsilon > 0$ there exist $\beta_1, \beta_2 \in (z_0(\alpha) - \varepsilon, z_0(\alpha) + \varepsilon)$ so that $y(T, \alpha, \beta_1)y(T, \alpha, \beta_2) < 0$.

From (A_2) we obtain

(A_3) for every $\varepsilon < 0$ and $y_0 \in [\alpha_1, \alpha_2]$, exists $\delta(\varepsilon) > 0$ so that if $|\bar{y}_0 - y_0| < \delta(\varepsilon)$ it exists $z_0(\bar{y}_0)$ so that $|z_0(\bar{y}_0) - z_0(y_0)| < \varepsilon$.

We consider now the function

$$Z(\alpha) = z(T, \alpha, z_0(\alpha)) - z_0(\alpha), \quad \alpha \in [\alpha_1, \alpha_2];$$

for this function we have $Z(\alpha_1) < 0$ and $Z(\alpha_2) > 0$. Then, as consequence of the continuity of the function $z(T, y_0, z_0)$ and the property (A_3) , there exists $\alpha_0 \in [\alpha_1, \alpha_2]$ so that $Z(\alpha_0) = 0$ or similarly $z(T, \alpha_0, z_0(\alpha_0)) - z_0(\alpha_0) = 0$. If we choose now $y_0 = \alpha_0$ and $z_0 = z_0(\alpha_0)$, we have $y(T, \alpha_0, z_0(\alpha_0)) = \alpha_0$, $z(T, \alpha_0, z_0(\alpha_0)) = z_0(\alpha_0)$ and so the Theorem 1 is proved.

The proof of the Theorem 2 is immediat if one takes into account the Theorem 1.

Remark. If a, b, c, d from the system (1) are constants the condition (H_2) can be satisfied if $T > 0$ is eventually sufficiently small.

It results that the system (1) with a, b, c, d constants has periodic solutions. Obviously, this is a less general conclusion as that which results by considering a direct proof when a, b, c, d are constants, as it is proved in [5].

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SOLUȚII PERIODICE PENTRU UN SISTEM VOLTERRA

(Rezumat)

Se consideră sisteme de forma (1) care descriu interdependența dintre o specie răpitoare și una nerăpitoare care trăiesc într-un același mediu și se determină mai întâi condiții în care există valori inițiale y_0, z_0 ale lui y și z și un timp T astfel încât să aibă loc (2); apoi se obțin sisteme de forma (1) cu coeficienții depinzând de timp care admit soluții T -periodice.

LA CIRCULATION ADMISSIBLE DANS UN GRAPHE FEUILLE

PAR

VIOREL GHIUȘ

Soit un graphe nonorienté $G=[N, A]$, où $N \neq \emptyset$ est l'ensemble des sommets et $A \neq \emptyset$ est l'ensemble des arêtes. On définit les fonctions numériques non négatifs $l(x, y)$ et $L(x, y)$ avec $l(x, y) \leq L(x, y)$ quelque soit $(x, y) \in A$, sur l'ensemble des arêtes.

Définition 1. On appelle *réseau non orienté* le triplet $R=(G, l, L)$.

Définition 2. On appelle une *orientation* de $G(R)$, une fonction $f: A \rightarrow N$ qui attribue à chaque arête l'une de ses extrémités, nommée *extrémité positive*.

Pour une orientation f de $G(R)$, on notera avec $G^f(R^f)$, le graphe (réseau) orienté attaché.

Soit $g(x)$ le degré local du sommet $x \in N$ du graphe non orienté G , $g'(x)$ le nombre des arêtes qui sortent du sommet x et $g''(x)$ le nombre des arêtes qui entrent dans le sommet x dans le graphe orienté attaché G^f .

Définition 3. On appelle une *circulation* dans le réseau non orienté, une paire (f, F) , où f est une orientation de G avec la propriété que les degrés locaux $g'(x) \geq 1$ et $g''(x) \geq 1$ pour toute $x \in N$ et F une circulation dans G^f , c'est à dire on satisfait les équations de conservation [1]

$$\sum (F(x, y); y \in N) - \sum (F(y, x); y \in N) = 0, \quad \forall x \in N.$$

Définition 4. On dit qu'une circulation est *admissible* si elle satisfait la condition

$$l(x, y) \leq F(x, y) \leq L(x, y), \quad \forall (x, y) \in A.$$

On sait [4] qu'un arête circuit doit appartenir au moins à un circuit et qu'un graphe feuille est constitué seulement des arêtes circuit.

On dit aussi, qu'une paire d'arêtes est *fortement connexe*, si elle appartient au même circuit et qu'un *graphe lobe* est formée seulement des paires d'arêtes fortement connexes.

Conformément aux Théorèmes 5.4.1-5.4.4 [4], tout graphe feuille G admet une décomposition de sous-graphes lobe maximaux

$$G = \sum (G_i; 1 \leq i \leq n),$$

tous deux sous-graphes lobe G_i , ayant en commun tout au plus un sommet.

Définition 5. La décomposition du graphe feuille dans les sous-graphes lobe s'appelle *graphe cactus*.

Théorème. La condition nécessaire et suffisante que dans un réseau cactus existe une circulation admissible est que dans chaque réseau lobe $G_i (i=\overline{1, n})$ on a une circulation admissible.

Nécessité. Considérons un sommet O de réseau de lobes composants de R ; soit tous les graphes lobes incidents dans O (Fig. 1)

$$\cup (G_i; 1 \leq i \leq n),$$

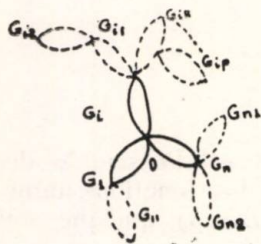


Fig. 1

où $G_i = (N_i, M_i)$ avec $i = \overline{1, n}$, qui a la propriété $\cap (G_i; 1 \leq i \leq n) = \{O\}$.

Supposons que dans le réseau cactus donné il existe une circulation admissible (f, F) ; l'orientation f de tous les arêtes de G^f induit une orientation de G_i ; $f_i(M_i) = f(M_i)$, c'est à dire qu'on obtient les sous-graphes orientés attachés G_i^f avec $i = \overline{1, n}$.

De l'existence de la circulation F au G^f nous avons

$$F(x, N) = F(N, x), \quad \forall x \in N.$$

On considère sous-arbre de lobes $G_i \cup (G_{ik}; 1 \leq k \leq p)$ qui éloigné du graphe cactus n'affecte pas la connexion; soit $G_{ik} = (N_{ik}, M_{ik})$ et on additionne les dernières équations d'après tous les sommets de la $\cup (N_{ik}; 1 \leq k \leq p) \cup (N_i - \{O\})$, $i = \overline{1, n}$.

Tout arc (a, b) figure dans les équations de conservation deux fois: une fois dans l'équation de conservation correspondante au sommet a et une fois quand on écrit l'équation de conservation pour le sommet b . On réduira donc tous les termes à l'exception de ceux correspondants aux arcs de M_i^f , incidents dans le sommets O : $F(N_i, O) = F(O, N_i)$, $\forall i = \overline{1, n}$, ce qui constitue l'équation de conservation du sommet O considéré en R_i ; on mentionne que $(N_i, O) \neq \emptyset$ et $(O, N_i) \neq \emptyset$ parce que les degrés locaux du sommet O en G et G_i ($i = \overline{1, n}$) sont $g(O) \geq 4$, $g_i(O) \geq 2$ ce qui implique $g_i'(O) \geq 1$ et $g_i''(O) \geq 1$, pour $i = \overline{1, n}$.

Les conditions admissibles du R^f étant en particulier respectés aussi en R_i^f , $i = \overline{1, n}$, résulte l'existence de circulations admissibles en R_i^f , $i = \overline{1, n}$.

Suffisance. S'il y a les circulations admissibles (f_i, F_i) dans les sous-réseaux R_i^f ($i = \overline{1, n}$), nous définissons circulation admissible (f, F) dans le réseau R :

$$\text{et} \quad \begin{aligned} f(M_i) &= f_i(M_i), \\ F(M_i) &= F_i(M_i), \end{aligned} \quad \text{pour} \quad i = \overline{1, n}.$$

Les conditions admissibles sont évidemment satisfaites en R^f aussi.

Les équations de conservation des $\cup (R_i^f; 1 \leq i \leq n)$ sont évidemment satisfaites pour tous les sommets différents du sommet O ; recherchons l'équation de conservation correspondante au sommet O :

$$F(O, \cup(N_i; 1 \leq i \leq n)) = \sum(F(O, N_i); 1 \leq i \leq n) = \sum(F_i(O, N_i); 1 \leq i \leq n) = \\ = \sum(F_i(N_i, O); 1 \leq i \leq n) = F(\cup(N_i; 1 \leq i \leq n), O).$$

Il existe donc une circulation admissible dans le réseau R .

Remarque. Ce théorème impose l'étude du problème de la circulation admissible seulement dans la catégorie des graphes lobes.

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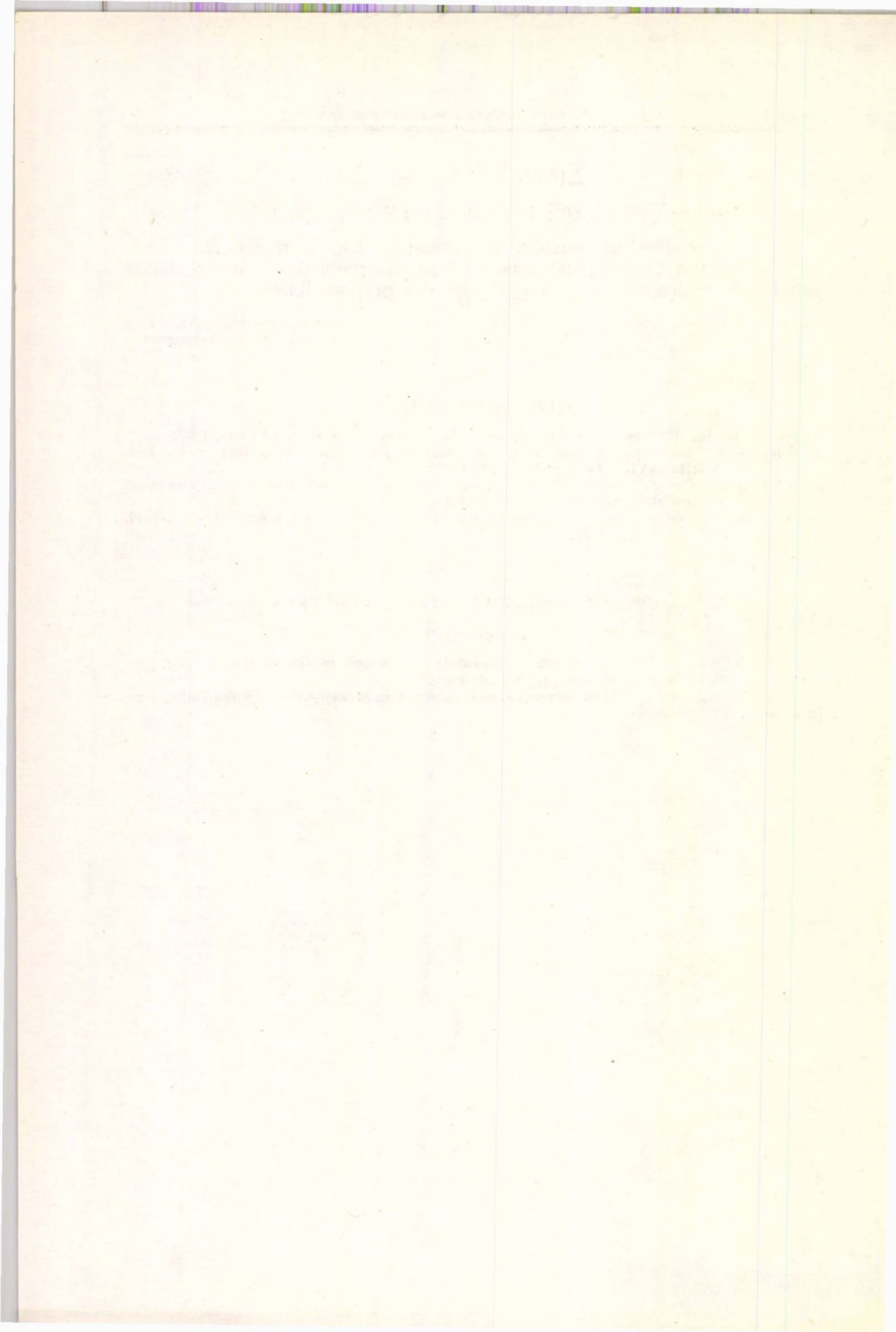
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CIRCULAȚIE ADMISIBILĂ ÎNTR-UN GRAF FRUNZĂ

(Rezumat)

Se definește noțiunea de circulație admisibilă într-un graf neorientat și se studiază problema circulației admisibile într-un graf frunză (cactus).

Se demonstrează o condiție necesară și suficientă pentru existența unei circulații admisibile în această categorie de grafuri.



SUR L'INVARIABILITÉ DE L'AXE INSTANTANÉ DE ROTATION

IV. LE CAS DES DEUX REPÈRES CARTÉSIENS TRIORTHOGONAUX, L'UN DE CES DEUX REPÈRES SE MOUVANT EXCLUSIVEMENT PAR UNE ROTATION, PAR RAPPORT À L'AUTRE

PLUSIEURS CONDITIONS SUFFISANTES POUR L'INVARIABILITÉ DE L'AXE INSTANTANÉ DE ROTATION

PAR

ELENA ACKER et C. ACKER

Lorsque l'origine O_2 (v. la Fig. 1), d'un repère mobile cartésien triorthogonal $X_2O_2Y_2Z_2$, est fixe par rapport au repère mobile cartésien triorthogonal $X_1O_1Y_1Z_1$ et $\mu \neq 1$, alors — tel que nous l'avons démontré dans [3] — la coïncidence des axes instantanés de rotation (Δ_2) respectif (Δ_1) , correspondant aux mouvements des repères $X_2O_2Y_2Z_2$ respectif $X_1O_1Y_1Z_1$, par rapport au repère fixe $XOYZ$, implique d'une manière claire que les relations

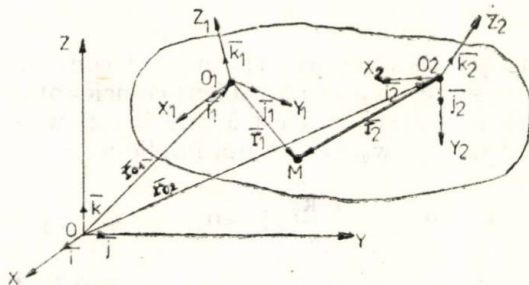


Fig. 1

(1)

$$w_2 = \mu w_1,$$

(2)

$$\left(1 - \frac{1}{\mu}\right) \left(\overrightarrow{O_1O_2} - \frac{w_1 \times v_{O_1}}{w_1^2} \right) \times w_1 = 0,$$

ou

(1)

$$w_2 = \mu w_1,$$

(3)

$$\left(1 - \frac{1}{\mu}\right) \left(\overrightarrow{O_1O_2} - \frac{w_1 \times v_{O_1}}{w_1^2} \right) = v w_1, \quad (v \in \mathbb{R}),$$

doivent être satisfaites.

Observation. La relation (2) (ou (3)) exprime: (α) le passage permanent par O_2 de l'axe instantané de rotation, correspondant au mouvement du repère $X_1O_1Y_1Z_1$ par rapport au repère fixe (ce qui implique l'appartenance permanente de M_1 à la sphère ayant O_1O_2 pour diamètre, M_1 étant la pointe du vecteur

$$(4) \quad \mathbf{W} = \frac{\mathbf{w}_1 \times \mathbf{v}_{O_1}}{w_1^2},$$

dont le point d'application est l'origine O_1 (v. Fig. 2 ; donc (2) (ou (3)) implique une position du pied M_1 de la perpendiculaire qui descend de O_1 sur l'axe instantané de rotation (Δ_1), qui doit appartenir à la sphère précisée au-dessus).

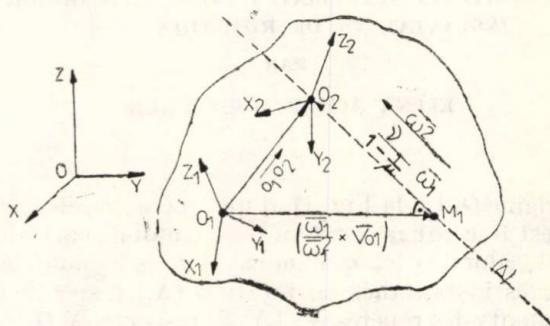


Fig. 2

Remarquons que les relations (1) et $\mu=1$ constituent une condition suffisante pour que les axes (Δ_1) et (Δ_2) soient coïncidents, mais dans ce cas il y a une situation plus spéciale, c'est à dire il suit $w_1=w_2$, qui implique (v. (1) et (2) de [3]) $w_{12}=-w_{21}=0$, ce qui implique

$$(5) \quad \frac{\partial^* \mathbf{j}_2}{\partial t} \cdot \mathbf{k}_2 = 0, \quad \frac{\partial^* \mathbf{k}_2}{\partial t} \cdot \mathbf{i}_2 = 0, \quad \frac{\partial^* \mathbf{i}_2}{\partial t} \cdot \mathbf{j}_2 = 0;$$

alors en appliquant la dérivée relative au repère $X_1O_1Y_1Z_1$, par rapport au temps t , aux deux membres des relations (évidemment satisfaites)

$$(6) \quad \mathbf{i}_2^2 = 1, \quad \mathbf{j}_2^2 = 1, \quad \mathbf{k}_2^2 = 1,$$

et

$$(7) \quad \mathbf{i}_2 \cdot \mathbf{j}_2 = 0, \quad \mathbf{j}_2 \cdot \mathbf{k}_2 = 0, \quad \mathbf{k}_2 \cdot \mathbf{i}_2 = 0,$$

on obtient

$$(8) \quad \mathbf{i}_2 \cdot \frac{\partial^* \mathbf{i}_2}{\partial t} = 0, \quad \mathbf{j}_2 \cdot \frac{\partial^* \mathbf{j}_2}{\partial t} = 0, \quad \mathbf{k}_2 \cdot \frac{\partial^* \mathbf{k}_2}{\partial t} = 0,$$

$$(9) \quad \frac{\partial^* \mathbf{i}_2}{\partial t} \cdot \mathbf{j}_2 = -\mathbf{i}_2 \cdot \frac{\partial^* \mathbf{j}_2}{\partial t}, \quad \frac{\partial^* \mathbf{j}_2}{\partial t} \cdot \mathbf{k}_2 = -\mathbf{j}_2 \cdot \frac{\partial^* \mathbf{k}_2}{\partial t}, \quad \frac{\partial^* \mathbf{k}_2}{\partial t} \cdot \mathbf{i}_2 = -\mathbf{k}_2 \cdot \frac{\partial^* \mathbf{i}_2}{\partial t},$$

ce qui implique (v. aussi (5))

$$(10) \quad \begin{cases} \frac{\partial^* \mathbf{i}_2}{\partial t} = \left(\frac{\partial^* \mathbf{i}_2}{\partial t} \cdot \mathbf{i}_2 \right) \mathbf{i}_2 + \left(\frac{\partial^* \mathbf{i}_2}{\partial t} \cdot \mathbf{j}_2 \right) \mathbf{j}_2 + \left(\frac{\partial^* \mathbf{i}_2}{\partial t} \cdot \mathbf{k}_2 \right) \mathbf{k}_2 = 0, \\ \frac{\partial^* \mathbf{j}_2}{\partial t} = \left(\frac{\partial^* \mathbf{j}_2}{\partial t} \cdot \mathbf{i}_2 \right) \mathbf{i}_2 + \left(\frac{\partial^* \mathbf{j}_2}{\partial t} \cdot \mathbf{j}_2 \right) \mathbf{j}_2 + \left(\frac{\partial^* \mathbf{j}_2}{\partial t} \cdot \mathbf{k}_2 \right) \mathbf{k}_2 = 0, \\ \frac{\partial^* \mathbf{k}_2}{\partial t} = \left(\frac{\partial^* \mathbf{k}_2}{\partial t} \cdot \mathbf{i}_2 \right) \mathbf{i}_2 + \left(\frac{\partial^* \mathbf{k}_2}{\partial t} \cdot \mathbf{j}_2 \right) \mathbf{j}_2 + \left(\frac{\partial^* \mathbf{k}_2}{\partial t} \cdot \mathbf{k}_2 \right) \mathbf{k}_2 = 0, \end{cases}$$

c'est à dire les directions des axes du repère $X_2O_2Y_2Z_2$, par rapport au repère $X_1O_1Y_1Z_1$ (qui a été considéré solidaire d'un corps rigide), restent invariables pendant l'écoulement du temps t .

Donc (1) et $\mu=1$, c'est à dire $\mathbf{w}_1 = \mathbf{w}_2$, implique une exclusive translation du repère $X_2O_2Y_2Z_2$, par rapport au repère $X_1O_1Y_1Z_1$, ce qui relève la relation $\mathbf{w}_1 = \mathbf{w}_2$ comme une condition nécessaire et suffisante pour que le mouvement du repère $X_2O_2Y_2Z_2$, par rapport au repère $X_1O_1Y_1Z_1$, soit une exclusive translation (v. aussi (9), (10), (11) de la 2-ème partie de cette étude [2], où nous avons démontré la nécessité de cette condition, la suffisance de cette condition n'étant pas nécessaire à être relevée, jusqu'ici, d'une manière pressante).

Seulement, ici, la vitesse commune, par rapport au repère $X_1O_1Y_1Z_1$, de tous les points solidaires du repère $X_2O_2Y_2Z_2$ (propriété caractéristique pour le mouvement de translation de $X_2O_2Y_2Z_2$ par rapport au repère $X_1O_1Y_1Z_1$), a toujours la valeur nulle, car O_2 , qui est solidaire du repère $X_2O_2Y_2Z_2$, a une position fixe par rapport au repère $X_1O_1Y_1Z_1$.

En conclusion, dans l'hypothèse constituée par (1) et $\mu=1$ (*), les repères $X_1O_1Y_1Z_1$, $X_2O_2Y_2Z_2$, ont vraiment le même axe instantané de rotation, parce que la condition (2) (ou (3)) est aussi satisfaite, mais de plus, les deux axes instantanés de rotation sont coïncidents parce que les deux repères mobiles sont devenus solidaires l'un de l'autre (v. la première partie de cette étude [1]).

C'est pour ça que, désormais, nous allons considérer la condition (1) écrite exclusivement pour $\mu \neq 1$.

Théorème. Lorsque l'origine O_2 du repère cartésien triorthogonal $X_2O_2Y_2Z_2$ est fixe par rapport au repère cartésien triorthogonal $X_1O_1Y_1Z_1$ (et $\mu \neq 1$), alors la condition nécessaire et suffisante pour que les axes instantanés de rotation des deux repères cartésiens triorthogonaux coïncident est exprimée par les relations (1) (2) (ou par les relations (1), (3)) (v. aussi [3]).

Vraiment, en imposant la condition (1), l'équation vectorielle de l'axe instantané de rotation (Δ_2) du repère $X_2O_2Y_2Z_2$, correspondante au mouvement de ce repère par rapport au repère fixe et rapportée au repère fixe

$$(*) \quad \mathbf{r} = \mathbf{r}_{O_2} + \frac{\mathbf{w}_2 \times \mathbf{v}_{O_2}}{\mathbf{w}_2^2} + \lambda \mathbf{w}_2$$

* On suppose toujours $\mathbf{w}_1, \mathbf{w}_2 \neq 0$, pour pouvoir considérer les axes instantanés de rotation (Δ_1), (Δ_2).

prend la forme (tel que nous l'avons démontré dans [3])

$$(11) \quad \mathbf{r} = \mathbf{r}_{O_1} + \frac{\mathbf{w}_1 \times \mathbf{v}_{O_1}}{w_1^2} + \left(1 - \frac{1}{\mu}\right) \left(\overrightarrow{O_1 O_2} - \frac{\mathbf{w}_1 \times \mathbf{v}_{O_1}}{w_1^2} \right) + \left[\frac{1}{\mu} \left(\frac{\mathbf{w}_1}{w_1^2} \cdot \overrightarrow{O_1 O_2} \right) + \lambda \mu \right] \mathbf{w}_1$$

et en imposant, de plus, la condition (2), ou (3), alors l'équation (*) représente aussi l'équation vectorielle de l'axe instantané de rotation (Δ_1) du repère $X_1 O_1 Y_1 Z_1$, donc les deux axes instantanés de rotation (Δ_1) , (Δ_2) coïncident.

Réciproquement, lorsque (Δ_1) , (Δ_2) coïncident, alors ils ont la même direction, donc la relation (1) est satisfaite et donc l'équation vectorielle, par rapport au repère fixe, de l'axe (Δ_2) , prend la forme (11); considérons alors aussi le point N , dont le vecteur de sa position, par rapport au repère fixe, est

$$\mathbf{P} = \mathbf{r}_{O_1} + \frac{\mathbf{w}_1 \times \mathbf{v}_{O_1}}{w_1^2}$$

(nous avons considéré ce vecteur aussi dans la 2-ème partie de cette étude); lorsque les axes (Δ_1) et (Δ_2) coïncident, alors le point N est l'un des points communs appartenant à ces deux axes (v. (4) de la première partie de cette étude); alors, pour obtenir le vecteur $\mathbf{r}$ (de (11) impliquée par (1)) de la position d'un point quelconque *de l'unique axe instantané de rotation*, on doit sommer le vecteur $\mathbf{P}$ à un vecteur parallèle à $\mathbf{w}_1$, ce qui entraîne aussi (v. (11)) soit (2), soit (3) (q.e.d.).

Observations. 1. L'équivalence des conditions (α) et (2) (ou (3)), permet de substituer (2) (ou (3)) par (α) , dans le texte du théorème précédent.

2. Le caractère réciproque pour les deux repères $X_1 O_1 Y_1 Z_1$, $X_2 O_2 Y_2 Z_2$: a) de la condition qui consiste en ce que les deux repères $X_1 O_1 Y_1 Z_1$, $X_2 O_2 Y_2 Z_2$ ont un point commun O_2 solidarisé à eux (donc un point fixe par rapport à ces deux repères mobiles) et b) de la condition (1), implique, sous ces deux conditions (a), (b), la suffisance (et la nécessité) du passage par O_2 d'un seul axe de la paire (Δ_1) , (Δ_2) , pour la coïncidence de ces deux axes.

Signalons encore le caractère suffisant des conditions (1) et

$$(12) \quad \overrightarrow{O_1 O_2} = \frac{\mathbf{w}_1 \times \mathbf{v}_{O_1}}{w_1^2}$$

pour la coïncidence des deux axes instantanés de rotation (Δ_1) , (Δ_2) , la condition (1) et la coïncidence des points M_1 et O_2 impliquant la coïncidence des deux axes instantanés de rotation, tel que l'unique axe instantané de rotation passe toujours par O_2 et il est toujours perpendiculaire sur $\overrightarrow{O_1 O_2}$ (sous la condition (12), O_2 est devenu le pied de la perpendiculaire qui descend de O_1 sur l'axe instantané de rotation du repère $X_1 O_1 Y_1 Z_1$, correspondant au mouvement de ce repère par rapport au repère fixe; v. la Fig. 2).

Enfin, voici de plus, deux cas spéciaux dans lesquels les axes instantanés de rotation des repères $X_1 O_1 Y_1 Z_1$, $X_2 O_2 Y_2 Z_2$, correspondants aux mouvements de ces repères par rapport au repère fixe, coïncident, en supposant toujours satisfaite la relation (1) (aussi que $\mu \neq 1$, sans exception).

A) Lorsque

$$(13) \quad \mathbf{v}_{O_1} = m \mathbf{w}_1, \quad (m \in \mathbb{R}),$$

c'est à dire si l'axe instantané de rotation (Δ_1) , du repère $X_1O_1Y_1Z_1$, passe par O_1 , alors il est nécessaire et suffisant que

$$(14) \quad \overrightarrow{O_1O_2} = n \mathbf{w}_1, \quad (n \in \mathbb{R}),$$

(v. (11)), c'est à dire il est nécessaire et suffisant que (Δ_1) passe aussi par O_2 , pour que les axes instantanés de rotation, des deux repères mobiles, soient coïncidents.

B) Lorsque $\overrightarrow{O_1O_2} = p \mathbf{w}_1$, $(p \in \mathbb{R})$, alors il est nécessaire et suffisant que $\mathbf{v}_{O_1} = q \mathbf{w}_1$, $(q \in \mathbb{R})$, pour que les axes instantanés de rotation (Δ_1) , (Δ_2) soient coïncidents.

Observons que dans les cas (A), (B) d'au-dessus, les axes instantanés de rotation coïncidents (Δ_1) , (Δ_2) , des repères mobiles $X_1O_1Y_1Z_1$, $X_2O_2Y_2Z_2$, passent par O_1 et par O_2 .

Relatif à ces deux cas spéciaux (A), (B), relevons le suivant

T h é o r è m e. Lorsque O_2 a une position fixe par rapport au repère mobile $X_1O_1Y_1Z_1$, et en supposant toujours satisfaite la relation (1) (aussi que $\mu \neq 1$), alors deux des trois propositions p_1 , p_2 , p_3 ci-dessous entraîne la troisième

$$p_1) \quad \mathbf{v}_{O_1} \parallel \mathbf{w}_1, \quad p_2) \quad \overrightarrow{O_1O_2} \parallel \mathbf{w}_1, \quad p_3) \quad (\Delta_1) \text{ et } (\Delta_2) \text{ coïncident.}$$

Pour faire la démonstration, v. (11) impliquée par (1).

Remarquons aussi, d'une manière semblable à celle de la 2-ème partie de la présente étude, le caractère restrictif, de plus en plus accentué, de la suite des conditions visant — pendant la substitution du repère $X_1O_1Y_1Z_1$ par $X_2O_2Y_2Z_2$, ou vice versa — l'invariabilité „instantanée“, „temporaire“ („intermittente“), ou „permanente“ de l'axe instantané de rotation de ces repères, par rapport au repère fixe.

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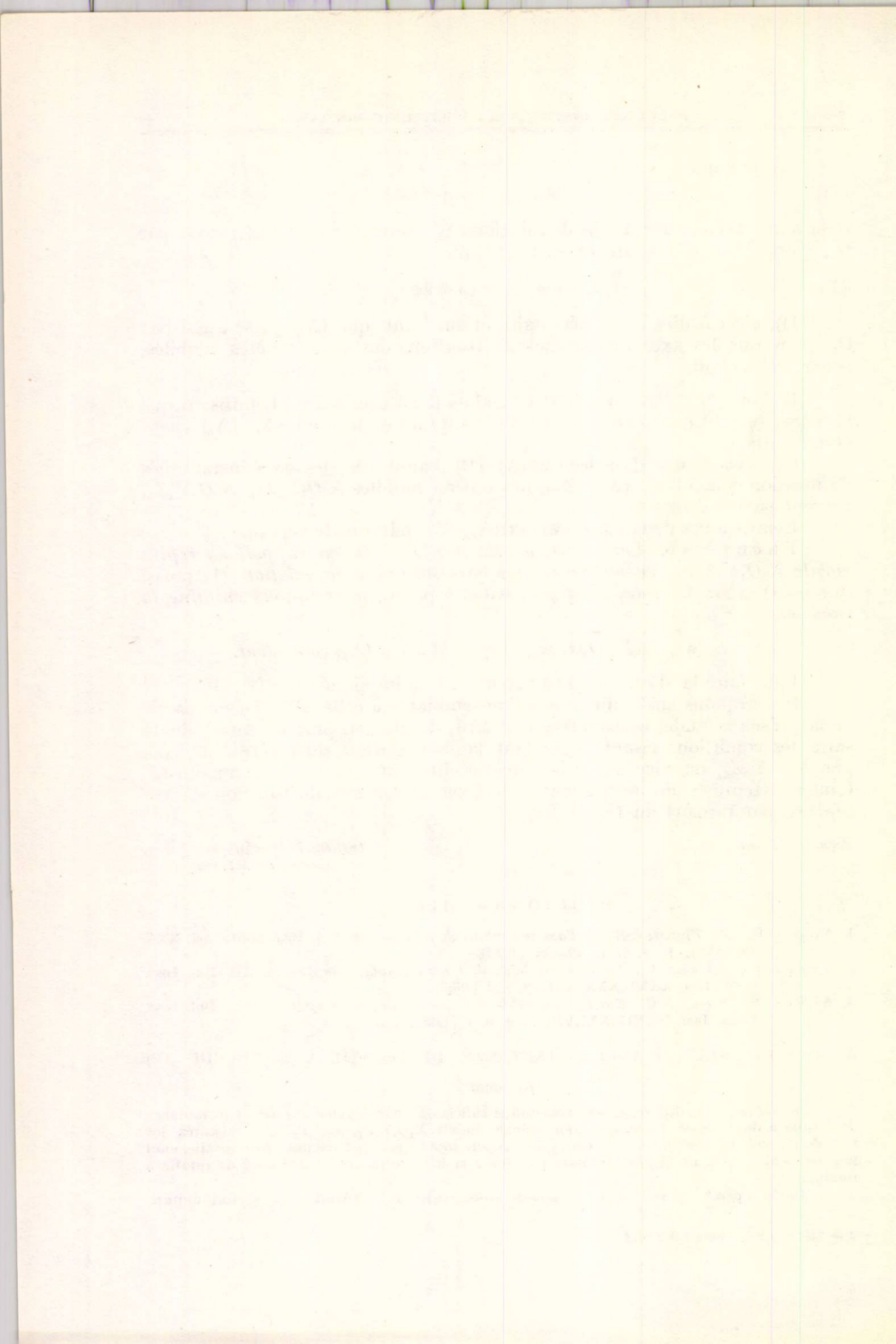
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ASUPRA INVARIANȚEI AXEI INSTANTANEE DE rotație A RIGIDULUI (IV)

(Rezumat)

Se stabilește condiția generală, necesară și suficientă, de coincidență a axelor instantanee de rotație a două repere carteziane triortogonale mobile $X_1O_1Y_1Z_1$, $X_2O_2Y_2Z_2$, în mișcarea lor față de reperul fix, pentru cazul în care cele 2 repere mobile se mișcă exclusiv prin rotație, unul față de celălalt, precum și alte rezultate privitoare la invarianța axei instantanee de rotație a rigidului.

Toate reperele carteziane triortogonale considerate sint alcătuite din vectori unitari.



ACCELERATION WAVES IN NONLINEAR COSSERAT MEDIA

BY

I. NISTOR

1. Introduction. The mathematical theory of acceleration waves in various materials has been greatly developed in recent years. Two related approaches have been used to study the theory of propagating singular surfaces. On the one hand, the concepts of displacement derivatives and compatibility conditions, as developed by Thomas [1] and the other hand the classical concepts of characteristic surface and bicharacteristic rays. The first method has been applied to acceleration waves in nonlinear elastic materials [2], in nonlinear thermoelastic materials [3] and in nonlinear hyperelastic Cosserat solids [4], [5]. The method of characteristic surfaces and bicharacteristic rays for totally hyperbolic linear equations is clearly stated in Courant and Hilbert [6]. Varley and Cumberbatch [7] showed that the method could be extended to hyperbolic quasilinear equations. This type of analysis has been applied to acceleration waves in nonlinear viscoelastic materials [8] and in simple elastic materials [9].

In this paper the bicharacteristic approach will be used to study acceleration waves in nonlinear theory of inhomogeneous, anisotropic, hyperelastic Cosserat materials. In the next section the basic formulae and equations of nonlinear theory of Cosserat media are given. In the following section the geometrical and kinematical conditions of compatibility are presented. In Sect. 4 the propagation condition and characteristic equation are derived. Next to be discussed are discontinuities on surfaces that correspond to multiple wave speeds. It turns out that growth or decay of discontinuities is controlled by a system of nonlinear, ordinary differential equations along each rays.

Explicit solutions are given for waves that correspond to double speeds of propagation and which propagate into a linear homogeneous material and the region ahead of the wave is in a state of uniform dilatation.

2. Equations of Cosserat Media. Throughout this paper a rectangular cartesian coordinate system is used. Lower case Latin minuscule characters are used for the suffixes of vector and tensor components, all having the range 1, 2, 3. The motion of a Cosserat body is described [10] by the functions

$$(2.1) \quad y_i = y_i(x_j, t), \quad R_{ij} = R_{ij}(x_k, t),$$

where y_i are the coordinates of the place occupied at time t of a particle that occupies the place of coordinates x_j in some reference configuration; R_{ij} are the components of a proper orthogonal tensor that characterizes the microrotation undergone by the body point.

The components of a proper orthogonal tensor may be written [11] in the form

$$(2.2) \quad R_{ij} = \frac{1}{1 + \frac{1}{4}\varphi_q\varphi_q} [(1 - \frac{1}{4}\varphi_k\varphi_k)\delta_{ij} + \frac{1}{2}\varphi_i\varphi_j + e_{jip}\varphi_p],$$

where φ_i are the components of the microrotation vector, e_{ijk} is the alternating symbol.

We consider [10] as strain tensors

$$(2.3) \quad c_{ij} = y_{k,i} = y_{k,i} R_{kj}, \quad \gamma_{ij} = \frac{1}{2} e_{jmn} R_{pn} R_{pm,i},$$

where the symbol $_i$ denotes differentiation with respect to the variable x_i .

The components v_i of the angular velocity vector are given [10] by

$$(2.4) \quad v_i = \frac{1}{2} e_{ijk} R_{jp} \dot{R}_{kp},$$

where a superposed dot indicates material differentiation.

Using (2.2) the relations (2.3)₂ and (2.4) may be represented in the form

$$(2.5) \quad \gamma_{ij} = H_{jp} \varphi_{p,i}, \quad v_i = H_{ji} \dot{\varphi}_j,$$

where

$$(2.6) \quad H_{jp} = \frac{2\delta_{jp} + e_{jpk}\varphi_k}{2(1 + \frac{1}{4}\varphi_i\varphi_i)}.$$

It is easy to verify that

$$(2.7) \quad H_{ki} = R_{ij} H_{jk} = H_{ij} R_{jk}.$$

From the relation (2.3)₂ and (2.4) results

$$(2.8) \quad R_{ij,k} = e_{jpq} R_{ip} H_{qn} \varphi_{n,k}, \quad \dot{R}_{ij} = e_{ipq} R_{qj} v_p.$$

The equations of motion of an anisotropic and inhomogeneous Cosserat media are [12]

$$(2.9) \quad \begin{aligned} (R_{ij} t_{kj})_{,k} + \rho F_i &= \rho \ddot{y}_i, \\ (R_{ij} m_{kj})_{,k} + e_{ijk} y_{j,p} R_{kp} t_{pq} + \rho G_i &= \rho \dot{\sigma}_i, \end{aligned}$$

where

$$(2.10) \quad t_{kj} = \frac{\partial W}{\partial c_{kj}}, \quad m_{kj} = \frac{\partial W}{\partial \gamma_{kj}}, \quad W = W(c_{ij}, \gamma_{pq}, x_k),$$

$$(2.11) \quad \sigma_p = J_{pq} v_q, \quad J_{pq} = R_{pk} R_{qh} I_{kh}.$$

In the above relations we have used the following notations: t_{ij} and m_{ij} — components of stress and couple-stress tensors measured per unit area of reference configuration; ρ — density of the body in reference configura-

tion; F_i — components of the body force vector; G_j — components of the body couple vector; σ_i — components of the microspin vector; I_{pq} — components of the material tensor of inertia density; W — strain energy per unit volume in the reference configuration.

Using relations (2.3), the Eqs. of motion (2.9) may be written in the form

$$(2.12) \quad \begin{aligned} t_{kp,k} + e_{pij}\gamma_{kj}t_{kj} + \rho F_i R_{ip} &= \rho R_{ip}\ddot{y}_i, \\ m_{kp,k} + e_{pij}(c_{ki}t_{kj} + \gamma_{ki}m_{kj}) + \rho R_{ip}G_i &= \rho R_{ip}\dot{\sigma}_i. \end{aligned}$$

3. Jump Conditions. Consider a regular surface Σ which is common boundary of the two regions $\mathcal{B}^+$ and $\mathcal{B}^-$ in any Cosserat body. Let $z(x_j, t)$ be a function which is continuous in the interiors of $\mathcal{B}^+$ and $\mathcal{B}^-$ and which approaches definite limit values z^+ and z^- when x_j approaches a point x_j^0 on Σ while remaining within $\mathcal{B}^+$ and $\mathcal{B}^-$, respectively. The jump of z across Σ at x_j^0 is denoted by

$$(3.1) \quad [z] = z^- - z^+.$$

If $[z] \neq 0$ the surface Σ is said to be singular with respect z . It is easy to verify the identity

$$(3.2) \quad [ab] = [a][b] + a^+[b] + b^+[a].$$

We consider a surface $\Sigma(t)$ given by a representation of the form $S(x_j, t) = 0$.

Components N_i of the unit normal to surface $\Sigma(t)$, in its direction of propagation (from the region $\mathcal{B}^-$ to the region $\mathcal{B}^+$) are given by

$$(3.3) \quad N_i = S_{,i}(S_{,j}S_{,j})^{-\frac{1}{2}}.$$

The speed of propagation V is defined by

$$(3.4) \quad V = -\dot{S}(S_{,j}S_{,j})^{-\frac{1}{2}}.$$

If $\dot{S}(x_j, t) \neq 0$, there is no loss in generality to assume that the surface may be represented by $t - \tau(x_j) = 0$. Then the following relations hold on $\Sigma(t)$:

$$(3.5) \quad N_i : V = \tau_{,i} : 1 = -S_{,i} : \dot{S}.$$

Let $(\cdot)$ denote a function evaluated on $\Sigma(t)$, that is

$$(3.6) \quad (z) = z(x_j, \tau(x_k)).$$

From the relations (3.1), (3.5) and (3.6) we have

$$(3.7) \quad (z)_{,i} = (z_{,i}) + (z)\frac{N_i}{V}, \quad [z]_{,i} = [z_{,i}] + \frac{N_i}{V}[z].$$

In particular case when $[z] = [\dot{z}] = [z_{,i}] = 0$, from the relations (3.7)₂ we obtain [9] the following jump conditions

$$(3.8) \quad [z_{,ij}] = -\frac{N_j}{V} [\dot{z}_{,i}] = \frac{N_i N_j}{V^2} [\ddot{z}],$$

$$(3.9) \quad [\dot{z}_{,ij}] = \frac{N_i N_j}{V^2} [\ddot{z}] - \frac{\partial}{\partial x_i} \left\{ \frac{N_j}{V} [\ddot{z}] \right\} - \frac{N_i}{V} [\ddot{z}]_{,j}.$$

A surface that is singular with respect to some quantity and that has a nonzero speed of propagation is said to be a propagating singular surface or wave.

A propagating singular surface $\Sigma(t)$ is said to be an acceleration wave in Cosserat media if the functions $y_i, \varphi_j, \dot{y}_i, \dot{\varphi}_j, \ddot{y}_i, \ddot{\varphi}_j$ are continuous everywhere, while the functions $\ddot{y}_i, \varphi_{i,jk}, \dot{y}_{j,ik}, \dot{y}_{i,j}, \varphi_{j,k}, \ddot{\varphi}_i$ have jump discontinuities across Σ , but are continuous functions everywhere else.

From the relations (2.3)–(2.5), (2.8), (3.8) and (3.9) we obtain for the case of acceleration waves in Cosserat media the following jump conditions

$$(3.10) \quad [\ddot{R}_{ij}] = e_{jpq} R_{ip} a_q^2, \quad [\dot{R}_{ij,k}] = -\frac{1}{V} e_{jpq} R_{ip} N_k a_q^2,$$

$$(3.11) \quad [R_{ij,pq}] = e_{jkh} R_{ih} a_k^2 \frac{N_p N_q}{V^2};$$

$$(3.12) \quad [c_{ij,k}^\alpha] = \frac{N_i N_k}{V^2} a_j^\alpha, \quad [\dot{c}_{ij}^\alpha] = -\frac{N_i}{V} a_j^\alpha, \quad (\alpha = 1, 2);$$

$$(3.13) \quad [\dot{v}_i] = R_{ij} a_j^2, \quad [\ddot{v}_i R_{ij}] = b_j^2 - \frac{1}{2} \dot{R}_{kj} R_{kp} a_p^2,$$

$$(3.14) \quad [\dot{c}_{ij,p}^\alpha] = \frac{N_i N_p}{V^2} b_j^\alpha - \frac{N_i}{V} a_{j,p}^\alpha - \frac{N_p}{V} a_{j,i}^\alpha - \left(\frac{N_p}{V} \right)_{,i} a_j^\alpha + B_{ijp}^\alpha, \quad (\alpha = 1, 2),$$

where

$$B_{ijp}^1 = e_{jkh} c_{ik} a_h^2 \frac{N_p}{V} + e_{jq} \gamma_{ik} a_q^1 + \dot{R}_{kj} R_{kp} a_q^1 \frac{N_i N_p}{V^2},$$

$$B_{ijp}^2 = e_{jkh} \gamma_{ik} a_h^2 \frac{N_p}{V} + \dot{R}_{kj} R_{kp} a_h^2 \frac{N_i N_p}{2V^2},$$

$$a_i^1 = R_{ji} [\ddot{y}_j], \quad a_i^2 = H_{ij} [\ddot{\varphi}_j], \quad b_j^1 = R_{kj} [\ddot{y}_k],$$

$$b_j^2 = \frac{1}{2} e_{jki} R_{hi} [\ddot{R}_{hk}], \quad c_{ij}^1 = c_{ij}, \quad c_{hk}^2 = \gamma_{hk}.$$

4. Amplitudes of Acceleration Waves. At points not on $\Sigma(t)$, the second and all higher derivatives of y_i and φ_j are assumed continuous. We assume that the external body force F_i , external body-couple G_j and the derivatives of the function $W(c_{ij}, \gamma_{pq}, x_k)$ with respect to x_i are continuous across the wave. The differential Eqs. (2.12) must hold on each side of the surface. On forming the jump of each term in Eqs. (2.12) on the singular surface $\Sigma(t)$ and using the compatibility conditions (3.12)₁ and (3.13)₁ we obtain

$$(4.1) \quad \begin{aligned} \frac{\partial^2 W}{\partial c_{kj} \partial c_{pq}^\alpha} N_p N_k a_q^\alpha &= \rho V^2 a_j^1, \\ \frac{\partial^2 W}{\partial \gamma_{kj} \partial c_{pq}^\alpha} N_p N_k a_q^\alpha &= \rho V^2 I_{jh} a_h^2. \end{aligned} \quad (\alpha = 1, 2),$$

In what follows Greek indices will have the range 1, 2, 3, 4, 5, 6. The vector $A = (A_\alpha)$ defined by $A_i = a_i^1$, $A_{i+3} = a_i^2$, is called the amplitude vector of the acceleration wave in Cosserat media.

We introduce a 6×6 matrix $E = (E_{\alpha\beta})$ and a 3×6 matrix $e = (e_{i\alpha})$ defined as follows

$$\begin{aligned} E_{pq} &= \delta_{pq}, & E_{p;q+3} &= 0, & E_{p+3;q} &= 0, & E_{p+3;q+3} &= I_{pq}, \\ e_{ij} &= c_{ij}, & e_{i;j+3} &= \gamma_{ij}. \end{aligned}$$

The Eqs. (4.1) may be written as

$$(4.2) \quad \left(\frac{\partial^2 W}{\partial e_{i\alpha} \partial e_{j\beta}} N_i N_j - \rho V^2 E_{\alpha\beta} \right) A_\beta = 0.$$

From (4.2) we see that the vector A can be non-trivial only if

$$(4.3) \quad \det \left(\frac{\partial^2 W}{\partial e_{i\alpha} \partial e_{j\beta}} N_i N_j - \rho V^2 E_{\alpha\beta} \right) = 0.$$

The Eq. (4.3) is a polynomial equation of degree six in V^2 , while this equation need not have any positive root, in the physical interpretation only real speeds and hence only positive roots can correspond to real acceleration waves.

It is known [4] that if the strain energy function W satisfies the condition

$$(4.4) \quad \frac{\partial^2 W}{\partial e_{i\alpha} \partial e_{j\beta}} N_i N_j \lambda_\alpha \lambda_\beta > 0,$$

for arbitrary non-vanishing vectors λ , then the roots of the Eq. (4.3) are positive.

The condition (4.4) ensures that there exist six speeds of propagation for an acceleration wave propagating in any desired direction (N_i).

If the strain energy satisfies the condition

$$(4.5) \quad \frac{\partial^2 W}{\partial e_{i\alpha} \partial e_{j\beta}} \mu_i \mu_j \lambda_\alpha \lambda_\beta > 0,$$

for arbitrary non-zero vectors μ and λ then there are six speeds of propagation for a wave front in any direction through (x_j) at time t .

Equation (4.3) may also be regarded as a first order partial equation for $\sum(t)$, hence a wave front is necessarily a characteristic surface.

We consider an acceleration wave propagating into a region which is undergoing a known deformation up to the arrival of the front. In such a region $V = h(N_i, x_j, t)$ can be determined from Eq. (4.3).

Bicharacteristics or rays and the variation of N_i and V along those rays are given [7] by the following ordinary differential equations

$$(4.6) \quad \frac{dx_i}{dt} = \frac{\partial h}{\partial N_i}, \quad \frac{dN_j}{dt} = (N_j N_k - \delta_{jk}) \frac{\partial h}{\partial x_k},$$

$$(4.7) \quad \frac{dV}{dt} = \frac{\partial h}{\partial t} + V N_i \frac{\partial h}{\partial x_i}.$$

Suppose that at initial time $t=0$ the wave front coincides with a surface which is represented parametrically as

$$(4.8) \quad x_i = x_i(u^1, u^2, 0),$$

so that normal at (u^1, u^2) to the wave front

$$(4.9) \quad N_i = N_i(u^1, u^2, 0)$$

is known.

The solution $x_i = x_i(u^1, u^2, t)$ of Eq. (4.6), which satisfies conditions (4.8) and (4.9) is the parametric representation of the front at time t .

To preserve uniformity of notation we may write $u^3 = t$, so that $x_j = x_j(u^1, u^2, u^3)$. Writing $J = \det(\partial x_j / \partial u^k)$, we have

$$(4.10) \quad \frac{dJ}{dt} = J \frac{\partial}{\partial x_i} \left(\frac{dx_i}{dt} \right).$$

If root V of Eq. (4.3) has multiplicity q ($q \leq 6$) and if $(f_{L\alpha})$, ($L=1, 2, \dots, q$) are any q linearly independent solutions of the system (4.2) corresponding to the speed V , then the amplitude vector of acceleration wave can be expressed as

$$(4.11) \quad A_\alpha = \psi_L f_{L\alpha}, \quad (L=1, 2, \dots, q),$$

where ψ_L are scalar factors.

Without loss of generality we can choose $(f_{L\alpha})$ so that

$$(4.12) \quad E_{\alpha\beta} f_{L\alpha} f_{K\beta} = \delta_{KL}, \quad (K, L=1, 2, \dots, q).$$

Since $(f_{L\alpha})$ is a solution of the system (4.2), are satisfied the following relations

$$(4.13) \quad \left(\frac{\partial^2 W}{\partial e_{k\alpha} \partial e_{p\beta}} N_p N_k - \rho V^2 E_{\alpha\beta} \right) f_{L\beta} = 0, \quad (L=1, 2, \dots, q).$$

By differentiating (4.13) with respect to N_i , at constant x_j and t , we get

$$(4.14) \quad \left(\frac{\partial^2 W}{\partial e_{i\alpha} \partial e_{p\beta}} N_p + \frac{\partial^2 W}{\partial e_{k\alpha} \partial e_{i\beta}} N_k - 2\rho \frac{\partial V}{\partial N_i} V E_{\alpha\beta} \right) f_{L\beta} + \\ + \left(\frac{\partial^2 W}{\partial e_{k\alpha} \partial e_{p\beta}} N_p N_k - \rho V^2 E_{\alpha\beta} \right) \frac{\partial f_{L\beta}}{\partial N_i} = 0.$$

We now multiply (4.14) by any vector $(f_{K\alpha})$ and making use of the relations (4.12) and (4.13) we arrive at

$$(4.15) \quad \left(\frac{\partial^2 W}{\partial e_{i\alpha} \partial e_{p\beta}} N_p + \frac{\partial^2 W}{\partial e_{k\alpha} \partial e_{i\beta}} N_k \right) f_{L\beta} f_{K\alpha} = 2\rho \frac{\partial V}{\partial N_i} V \delta_{KL}, \quad (K, L = 1, 2, \dots, q).$$

By taking the material derivative of the system (2.12) evaluating the jump of the resulting equations and using the relations (3.2), (3.10)–(3.14), (4.1) and (4.15) we obtain

$$(4.16) \quad 2\rho \frac{d\psi_L}{dt} + A_{LKP} \psi_K \psi_P + A_{LK} \psi_K = 0, \quad (L, K, P = 1, 2, \dots, q),$$

where

$$(4.17) \quad A_{LKP} = \frac{1}{V^3} \frac{\partial^3 W}{\partial e_{i\alpha} \partial e_{j\beta} \partial e_{k\delta}} N_i N_j N_k f_{L\alpha} f_{K\beta} f_{P\delta},$$

$$V A_{LK} = \tilde{B}_{LK} + D_{LK} - D_{KL},$$

$$(4.18) \quad \begin{aligned} \tilde{B}_{LK} = & \left\{ \frac{\partial^2 W}{\partial e_{k\beta} \partial e_{r\alpha}} \left[2f_{K\alpha}, ({}_k N_r) + \left(\frac{N_r}{V} \right)_{,x_i} V f_{K\alpha} \right] + \right. \\ & + \left[\frac{\partial^3 W}{\partial e_{k\alpha} \partial e_{p\beta} \partial e_{i\delta}} \left(e_{i\delta p}^+ - e_{i\delta}^+ \frac{N_p}{V} \right) + \frac{\partial^3 W}{\partial e_{i\beta} \partial e_{k\alpha} \partial x_i} \right] N_k f_{K\alpha} \Big\} f_{L\beta} + \\ & + [e_{pij} m_{kj} N_k + \rho V R_{hk} v_h (e_{pij} I_{jk} + e_{pk} t I_{ti})] f_{L;p+3} f_{K;i+3}, \end{aligned}$$

$$(4.19) \quad \begin{aligned} D_{LK} = & \left\{ \frac{\partial^2 W}{\partial c_{hj} \partial e_{k\alpha}} (f_{Lp} \gamma_{hi} + c_{hi} f_{L;p+3}) f_{K\alpha} + \right. \\ & + \left(\frac{\partial^2 W}{\partial \gamma_{rj} \partial e_{k\alpha}} \gamma_{hi} f_{K\alpha} + t_{kj} f_{Ki} \right) f_{L;p+3} \Big\} e_{pij} N_k. \end{aligned}$$

Parentheses inclosing two running indices indicate that the tensor is completely symmetrized with respect to the enclosed indices, for example $2a_{(ij)} = a_{ij} + a_{ji}$.

The Eqs. (4.16) are a system of q coupled, non-linear ordinary differential equations for the variation in strength of an acceleration front along bicharacteristic curves. The coefficients in the equations are in general functions of t , which can be determined if the motion of Cosserat medium is known in the region into which the front is spreading.

In the simplest case, the wave under consideration has a speed of propagation that is an isolated root of (4.2) then the jump in acceleration, as is given by (4.11), reduces to a single term $A_\alpha = \psi_L f_{L\alpha}$, (L fixed) and the system (4.16) reduces to a single equation

$$(4.20) \quad \frac{d}{dt} (\rho J \psi_L^2) + A_L^1 J \psi_L^2 + A_L^2 J \psi_L^3, \quad (L, \text{fixed})$$

where

$$A_L^1 = \rho R_{hk} v_h e_{pk} t I_{ti} f_{L;p+3} f_{L;i+3} + \frac{1}{V} \frac{\partial^3 W}{\partial e_{k\alpha} \partial e_{p\beta} \partial e_{i\delta}} \left(e_{i\delta p}^+ - \frac{N_p}{V} e_{i\delta}^+ \right) N_k f_{L\alpha} f_{L\beta},$$

A_L^2 is obtained from (4.17) for $K=L=P$ and J is given by (4.10).

The global behavior of the Eq. (4.20) has been discussed by Wright [9].

If the material is linear then $A_L^1 = A_L^2 = 0$ and the amplitude obeys the rule

$$\rho J \psi_L^2 = \text{const.}$$

This last result for particular case when the acceleration wave propagates into homogeneous materials where the region ahead of the wave is in a constant state of uniform deformation was given in [13].

The results obtained refer to an arbitrary elastic Cosserat solid. They involve no restriction upon either the form of singular surface or the state of the material ahead of it.

In particular case when the waves propagate into homogeneous linearly Cosserat materials Eqs. (4.16) become

$$(4.21) \quad \frac{d\psi_L}{dt} = B_{LK} \psi_K,$$

where

$$(4.22) \quad \begin{aligned} 2\rho B_{LK} = & - \frac{\partial^2 W}{\partial e_{j\beta} \partial e_{i\alpha}} \left[f_{K\alpha}, ({}_j N_i) \frac{2}{V} + \left(\frac{N_i}{V} \right)_{,j} f_{K\alpha} \right] f_{L\beta} + \\ & + \frac{\partial^2 W}{\partial c_{kj} \partial e_{i\alpha}} e_{pkj} (f_{L\alpha} f_{K;p+3} - f_{L;p+3} f_{K\alpha}) \frac{N_i}{V}. \end{aligned}$$

Note that if the singular surface propagates into a homogeneously strained region then function h does not depend explicitly on x_i and t . From the relations (4.6)₂ and (4.7) results that V and N_i are constant along a ray and hence B_{LK} given by (4.22) do not depend on t .

In the case of two-fold multiplicity, the jump in acceleration has the representation

$$(4.23) \quad A_\alpha = \psi_1 f_{1\alpha} + \psi_2 f_{2\alpha},$$

and the Eq. (4.21) reduces to

$$(4.24) \quad \frac{d\psi_1}{dt} = B_{11}\psi_1 + B_{12}\psi_2, \quad \frac{d\psi_2}{dt} = B_{21}\psi_1 + B_{22}\psi_2,$$

where B_{ij} , $(i, j=1, 2)$ is constant.

The solution to (4.24) may be determined by the standard method. There are two cases to consider:

i) $B_{12} = 0$:

For

$$(4.25) \quad B_{11} \neq B_{22}, \quad \psi_1 = C_1 (B_{11} - B_{22}) e^{B_{11}t}, \quad \psi_2 = C_2 e^{B_{22}t} + B_{21} C_1 e^{B_{11}t}.$$

For

$$(4.26) \quad B_{11} = B_{22}, \quad \psi_1 = C_1 e^{B_{11}t}, \quad \psi_2 = (C_2 + B_{21} C_1 t) e^{B_{11}t}.$$

ii) $B_{12} \neq 0$:

For $\Delta = (B_{11} - B_{22})^2 + 4B_{12}B_{21} > 0$,

$$(4.27) \quad \begin{aligned} \psi_1 &= B_{12}(C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}), \\ \psi_2 &= C_2(\lambda_1 - B_{11})e^{\lambda_1 t} + C_2(\lambda_2 - B_{11})e^{\lambda_2 t}, \end{aligned}$$

where

$$2\lambda_1 = B_{11} + B_{22} + \sqrt{\Delta}, \quad 2\lambda_2 = B_{11} + B_{22} - \sqrt{\Delta}.$$

For $\Delta = 0$,

$$(4.28) \quad \begin{aligned} \psi_1 &= 2B_{12}(C_2 + C_1 t)e^{2^{-1}(B_{11} + B_{22})t}, \\ \psi_2 &= [2C_1 + (B_{22} - B_{11})(C_1 t + C_2)]e^{2^{-1}(B_{11} + B_{22})t}. \end{aligned}$$

For $\Delta < 0$,

$$\begin{aligned} \psi_1 &= 2B_{12} \left(C_1 \cos \frac{\sqrt{-\Delta}t}{2} + C_2 \sin \frac{\sqrt{-\Delta}t}{2} \right) e^{2^{-1}(B_{11} + B_{22})t}, \\ 4.2 \quad 9) \quad \psi_2 &= \left[(B_{22} - B_{11}) \left(C_1 \cos \frac{\sqrt{-\Delta}t}{2} + C_2 \sin \frac{\sqrt{-\Delta}t}{2} \right) + \right. \\ &\quad \left. + \sqrt{-\Delta} \left(C_2 \cos \frac{\sqrt{-\Delta}t}{2} - C_1 \sin \frac{\sqrt{-\Delta}t}{2} \right) \right] e^{2^{-1}(B_{11} + B_{22})t}. \end{aligned}$$

In the relations (4.25)–(4.29) C_1 and C_2 are arbitrary real constants.

The magnitude ψ of the amplitude vector of acceleration wave under consideration is given by $\psi = \sqrt{\psi_1^2 + \psi_2^2}$.

When $B_{11} + B_{22} > 0$, the magnitude of amplitude vector increases becoming infinite, while $B_{11} + B_{22} < 0$ and $B_{11}B_{22} - B_{12}B_{21} > 0$ decreases approaching zero as $t \rightarrow \infty$.

Note if $B_{11}B_{22} - B_{12}B_{21} < 0$ amplitude of acceleration waves, given by (4.25) or (4.27), will become infinite monotonically as $t \rightarrow \infty$, but if $B_{11} + B_{22} < 0$ the magnitude ψ obtained from the relations (4.26), (4.28), (4.29) will decay to zero as $t \rightarrow \infty$. On the other hand if $B_{11} + B_{22} = 0$, then the magnitude obtained from (4.25)–(4.28), increases monotonically becoming infinite as $t \rightarrow \infty$, while the magnitude obtained from (4.29) neither grows nor decays.

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UNDE DE ACCELERAȚIE ÎN MEDII COSSERAT NELINIARE

(Rezumat)

Se studiază propagarea undelor de accelerație în medii Cosserat neliniare. Se demonstrează că variația amplitudinii undelor de accelerație de-a lungul bicaracteristicilor este guvernată de un sistem de ecuații diferențiale ordinare neliniare.

THERMAL STRESSES IN ELASTIC CYLINDERS WITH MICROSTRUCTURE

BY

SILVIA GABRIELA CIUMAȘU

In this paper we generalize the results of [1] to the case of elastic cylinders with microstructure. We consider the problem of thermal stresses in homogeneous isotropic and centrosymmetric elastic cylinders with microstructure when the temperature distribution is a polynomial of the r -th degree in the axial coordinate x_3 , namely

$$(1) \quad T = \sum_{k=0}^r T_k(x_1, x_2) x_3^k.$$

We assume that the functions $T_k(x_1, x_2)$ are given. The problem is solved using some results established in [2]. We observe that in this case can be defined two states of generalized plane strain.

Let V be the domain occupied by the beam limited by two planes $x_3=0$, $x_3=h$ and by a cylindrical surface B . The domain of cross-section of the beam and its area will be denoted S and the boundary of S is L . We assume that the body forces and the body couples are absent and the lateral surface is free of applied forces and couples. We assume the cylinder to be in equilibrium under the action of a given temperature, the loading applied to the ends being statically equivalent to zero.

The basic equations are [4]:

i) *Equilibrium equations*

$$(2) \quad \tau_{ij,i} + \sigma_{ij,i} = 0, \quad \mu_{ijk,i} + \sigma_{jk} = 0;$$

ii) *Constitutive equations*

$$(3) \quad \begin{aligned} \tau_{pq} &= \lambda \delta_{pq} \varepsilon_{ii} + 2\mu \varepsilon_{pq} + g_1 \delta_{pq} \gamma_{ii} + g_2 (\gamma_{pq} + \gamma_{qp}) + \beta_1 \delta_{pq} T, \\ \sigma_{pq} &= g_1 \delta_{pq} \varepsilon_{ii} + 2g_2 \varepsilon_{pq} + b_1 \delta_{pq} \gamma_{ii} + b_2 \gamma_{pq} + b_3 \gamma_{qp} + \beta_2 \delta_{pq} T, \\ \mu_{pqr} &= a_1 (\chi_{tip} \delta_{qr} + \chi_{rit} \delta_{pq}) + a_2 (\chi_{tiq} \delta_{pr} + \chi_{rit} \delta_{pq}) + a_3 \chi_{tir} \delta_{pq} + a_4 \chi_{piti} \delta_{qr} + \\ &\quad + a_5 (\chi_{qti} \delta_{pr} + \chi_{ipt} \delta_{qr}) + a_6 \chi_{iqi} \delta_{pr} + a_{10} \chi_{pqr} + a_{11} (\chi_{rpq} + \chi_{qrp}) + \\ &\quad + a_{13} \chi_{prq} + a_{14} \chi_{qpr} + a_{15} \chi_{rqr}; \end{aligned}$$

iii) *Geometrical equations*

$$(4) \quad \varepsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad \gamma_{ij} = u_{j,i} - \varphi_{ij}, \quad \chi_{ijk} = \varphi_{jki}.$$

In these relations we have used the notations from [4].

On the lateral surface of the cylinder we have the following conditions

$$(5) \quad (\tau_{\alpha i} + \sigma_{\alpha i})n_{\alpha} = 0, \quad \mu_{\alpha i} n_{\alpha} = 0, \quad \text{on } B,$$

where $(n_1, n_2, 0)$ are the direction cosines of the outward normal to the lateral surface.

On the plane $x_3 = 0$ we must have

$$(6) \quad \int_S (\tau_{3\alpha} + \sigma_{3\alpha}) d\sigma = 0, \quad \int_S [x_{\alpha} (\tau_{33} + \sigma_{33}) + \mu_{3\alpha 3} - \mu_{33\alpha}] d\sigma = 0,$$

$$\int_S (\tau_{33} + \sigma_{33}) d\sigma = 0, \quad \int_S e_{3\alpha\beta} [x_{\alpha} (\tau_{3\beta} + \sigma_{3\beta}) + \mu_{3\alpha\beta}] d\sigma = 0.$$

The problem consists in solving Eqs. (2)–(4), with conditions (5), (6) assuming that the function T has the form (1).

First we define two states of generalized plane strain of the considered cylinder. We consider (I), a first state of generalized plane strain of the cylinder to be that state in which

$$(7) \quad u_{\alpha} = u_{\alpha}(x_1, x_2), \quad \varphi_{\alpha\beta} = \varphi_{\alpha\beta}(x_1, x_2), \quad \varphi_{33} = \varphi_{33}(x_1, x_2), \quad u_3 = \varphi_{3\alpha} = \varphi_{\alpha 3} = 0.$$

The equilibrium equations, in this case are

$$(8) \quad (\tau_{\alpha\beta} + \sigma_{\alpha\beta})_{,\alpha} + f_{\beta} = 0, \quad \mu_{\alpha\beta\gamma,\alpha} + \sigma_{\beta\gamma} + L_{\beta\gamma} = 0, \quad \mu_{\alpha 33,\alpha} + \sigma_{33} + L_{33} = 0.$$

On the lateral surface of the cylinder we have the conditions

$$(9) \quad (\tau_{\alpha\beta} + \sigma_{\alpha\beta})n_{\alpha} = P_{\beta}, \quad \mu_{\alpha\beta\gamma} n_{\alpha} = Q_{\beta\gamma}, \quad \mu_{\alpha 33} n_{\alpha} = Q_{33}.$$

The necessary and sufficient conditions to solve the above problem are [2]

$$(10) \quad \int_L P_i ds + \int_S f_i d\sigma = 0, \quad \int_L e_{3\alpha\beta} (x_{\alpha} P_{\beta} + Q_{\alpha\beta}) ds + \int_S e_{3\alpha\beta} (x_{\alpha} f_{\beta} + L_{\alpha\beta}) d\sigma = 0.$$

We define (II), a second state of generalized plane strain, characterized by

$$(11) \quad u_3 = u_3(x_1, x_2), \quad \varphi_{3\alpha} = \varphi_{3\alpha}(x_1, x_2), \quad \varphi_{\alpha 3} = \varphi_{\alpha 3}(x_1, x_2), \quad u_{\alpha} = \varphi_{\alpha\beta} = \varphi_{\beta 3} = 0.$$

The equilibrium equation can be written in the form

$$(12) \quad (\tau_{\alpha 3} + \sigma_{\alpha 3})_{,\alpha} + f_3 = 0, \quad \mu_{\alpha\beta 3,\alpha} + \sigma_{\beta 3} + L_{\beta 3} = 0, \quad \mu_{\alpha 3\beta,\alpha} + \sigma_{3\beta} + L_{3\beta} = 0.$$

Let us assume that on the lateral surface of the cylinder we have the conditions

$$(13) \quad (\tau_{\alpha 3} + \sigma_{\alpha 3})n_{\alpha} = P_3, \quad \mu_{\alpha\beta 3} n_{\alpha} = Q_{\beta 3}, \quad \mu_{\alpha 3\beta} n_{\alpha} = Q_{3\beta}.$$

The conditions of static equilibrium for the cylinder in this case are [2

$$(14) \quad \int_S (x_2 f_3 + L_{23} - L_{32}) d\sigma + \int_L (x_2 P_3 + Q_{23} - Q_{32}) ds - \int_S (\tau_{32} + \sigma_{32}) d\sigma = 0,$$

$$\int_S (x_1 f_3 + L_{13} - L_{31}) d\sigma + \int_L (x_1 P_3 + Q_{13} - Q_{31}) ds - \int_S (\tau_{31} + \sigma_{31}) d\sigma = 0.$$

To solve the problem of thermal stresses in elastic cylinders with microstructure we use the method of induction. Let us denote by $P^{(n)}$ the problem of solving Eqs. (2)–(4) with conditions (5), (6) assuming that the function T has the form

$$(15) \quad T = T_n(x_1, x_2) x_3^n, \quad n \geq 0.$$

For $n=0$ we have the problem $P^{(0)}$. We suppose that the cylinder is subjected to the temperature distribution

$$T = f(x_1, x_2),$$

where $f(x_1, x_2)$ is a given function.

We try to solve the problem assuming that

$$(16) \quad u_\alpha = e_{\alpha\beta\gamma} (-\frac{1}{2} l_\beta x_\beta + l_\gamma x_\gamma) x_3 + \sum_{s=1}^4 l_s v_\alpha^{(s)} + v_\alpha(x_1, x_2),$$

$$u_3 = (e_{3\alpha\beta} x_\alpha l_\beta + l_3) x_3 + \sum_{s=1}^4 l_s v_3^{(s)} + v_3(x_1, x_2),$$

$$\varphi_{ij} = e_{jikh} l_k x_h + \sum_{s=1}^4 l_s \psi_{ij}^{(s)} + \psi_{ij}(x_1, x_2),$$

where $v_i(x_1, x_2)$, $\psi_{ij}(x_1, x_2)$ are unknown functions, l_s are unknown constants. The functions $v_\alpha^{(s)}$, $\psi_{\alpha\beta}^{(s)}$, $\psi_{33}^{(s)}$ ($s=1, 2, 4$) which appear in (16) are the solutions of three problems $A^{(s)}$ ($s=1, 2, 4$) of the type (I), and the functions $v_3^{(3)}$, $\psi_{3\alpha}^{(3)}$, $\psi_{\alpha 3}^{(3)}$ are the solutions of the problem $A^{(3)}$, of the type (II).

The problems $A^{(s)}$ ($s=1, 2, 3, 4$) correspond to the systems of loading $(f_i^{(s)}, L_{ij}^{(s)}, P_i^{(s)}, Q_{ij}^{(s)})$ where

$$(17) \quad \begin{aligned} f_\beta^{(8)} &= [\delta_{\alpha\beta} e_{3\rho\delta} x_\delta (\lambda + 2g_1 + b_1)]_{,\alpha}, & f_\beta^{(4)} &= [\delta_{\alpha\beta} (\lambda + 2g_1 + b_1)]_{,\alpha}, \\ f_3^{(3)} &= [e_{\alpha\beta\gamma} x_\beta (g_1 + 2g_2 + b_3)]_{,\alpha}, & f_3^{(8)} &= f_3^{(3)} = 0, \\ L_{\beta\gamma}^{(8)} &= \delta_{\beta\gamma} e_{3\rho\delta} x_\delta (g_1 + b_1), & L_{\beta\gamma}^{(4)} &= \delta_{\beta\gamma} (g_1 + b_1), \\ L_{33}^{(8)} &= e_{3\rho\delta} x_\delta (g_1 + 2g_2 + b_1 + b_2 + b_3), & L_{33}^{(3)} &= e_{\beta\alpha\gamma} x_\alpha (b_3 + g_2), \\ L_{33}^{(4)} &= g_1 + 2g_2 + b_1 + b_2 + b_3, & L_{33}^{(8)} &= e_{\beta\alpha\gamma} x_\alpha (b_2 + g_2), \\ L_{\beta\gamma}^{(3)} &= L_{33}^{(3)} = L_{\alpha 3}^{(4)} = L_{3\alpha}^{(4)} = L_{\beta\beta}^{(5)} = L_{3\beta}^{(5)} = 0, \\ P_\beta^{(8)} &= -\delta_{\alpha\beta} e_{3\rho\delta} x_\delta (\lambda + 2g_1 + b_1) n_\alpha, & P_4^{(4)} &= -\delta_{\alpha\beta} (\lambda + 2g_1 + b_1) n_\alpha, \end{aligned}$$

$$\begin{aligned}
P_3^{(3)} &= -e_{\alpha\beta\gamma} x_\beta (b_\gamma + 2g_\gamma + \mu) n_\alpha, & P_3^{(8)} &= P_\beta^{(3)} = P_3^{(4)} = 0, \\
Q_{qr}^{(s)} &= n_\alpha [a_1 \delta_{qr} e_{\alpha 3s} + a_2 (e_{q3s} \delta_{\alpha r} + e_{3rs} \delta_{\alpha q}) + a_3 \delta_{\alpha q} e_{r3s} + a_5 e_{3\alpha s} \delta_{qr} + \\
&\quad + a_8 e_{3qs} \delta_{\alpha r} + a_{11} (e_{q\alpha s} \delta_{r3} + e_{\alpha rs} \delta_{q3}) + a_{14} e_{r\alpha s} \delta_{32} + a_{15} e_{\alpha 2s} \delta_{r3}], \\
(s=1, 2, 3), & \quad Q_{qr}^{(4)} = 0.
\end{aligned}$$

It is easy to show that the necessary and sufficient conditions ((10), (14)) for the existence of the solution of the problems $A^{(s)}$ ($s=1, 2, 3, 4$) are satisfied. We assume that the functions $v_i^{(s)}, \psi_{ij}^{(s)}$ ($s=1, 2, 3, 4$) are known. Taking into account (16), (2)–(4) it follows that the functions $v_i(x_1, x_2)$, $\psi_{ij}(x_1, x_2)$ satisfy the equations of the generalized thermoelastic plane strain for the temperature distribution $T=f(x_1, x_2)$. From the conditions (6)_{2,3,4} we obtain the following system for the unknown constants

$$(18) \quad \sum_{s=1}^4 l_s D_{\alpha s} = N_1, \quad \sum_{s=1}^4 l_s D_{3s} = N_2, \quad \sum_{s=1}^4 l_s D_{4s} = N_3,$$

where we have used the notations

$$D_{\alpha s} = \int_S [x_\alpha (\tau_{33}^{(s)} + \sigma_{33}^{(s)} + \mu_{3\alpha 3}^{(s)} - \mu_{33\alpha}^{(s)})] d\sigma,$$

$$D_{3s} = \int_S (\tau_{33}^{(s)} + \sigma_{33}^{(s)}) d\sigma,$$

$$D_{4s} = \int_S e_{3\alpha\beta} [x_\alpha (\tau_{33}^{(s)} + \sigma_{33}^{(s)} + \mu_{3\alpha\beta}^{(s)})] d\sigma,$$

$$N_1 = - \int_S [x_\alpha (\tau_{33}^* + \sigma_{33}^*) + \mu_{3\alpha 3}^* - \mu_{33\alpha}^*] d\sigma,$$

$$N_2 = - \int_S (\tau_{33}^* + \sigma_{33}^*) d\sigma,$$

$$N_3 = - \int_S e_{3\alpha\beta} [x_\alpha (\tau_{33}^* + \sigma_{33}^*) + \mu_{3\alpha\beta}^*] d\sigma,$$

$\tau_{33}^*, \sigma_{33}^*, \tau_{3\beta}^*, \sigma_{3\beta}^*, \mu_{3\alpha 3}^*, \mu_{33\alpha}^*$ — are the components of the stress tensor and of the couple-stress tensor corresponding to $v_i(x_1, x_2)$, $\psi_{ij}(x_1, x_2)$. In the paper [2] was proved that $\det(D_{rs}) \neq 0$, so that the system (18) uniquely determines the constants l_s . Thus, the problem $P^{(s)}$ is solved.

In what follows we denote by u_i^*, φ_{ij}^* the components of the displacement vector and the microdeformation tensor from the problem $P^{(n)}$ and by u_i, φ_{ij} the analogous functions from the problem $P^{(n+1)}$.

The problem can be presented as follows: to find the functions u_i , φ_{ij} which satisfy the Eqs. (2)–(4) and the conditions (5), (6) when the temperature has the form

$$T = f(x_1, x_2) x_3^{(n+1)}$$

assuming that the functions u_i^* , φ_{ij}^* which satisfy the Eqs. (2)–(4) and the conditions (5), (6) for the temperature distribution

$$T = f(x_1, x_2) x_3^n,$$

are known.

We try to solve the problem assuming that

$$\begin{aligned} u_\alpha &= (n+1) \left\{ \int_0^{x_3} u_\alpha^* dx_3 + e_{\alpha\beta\gamma} \left(-\frac{1}{2} l_\beta x_\beta + l_\gamma x_\gamma \right) x_3 + \sum_{s=1}^4 l_s v_\alpha^{(s)} + w_\alpha(x_1, x_2) \right\}, \\ (19) \quad u_3 &= (n+1) \left\{ \int_0^{x_3} u_3^* dx_3 + (e_{3\alpha\beta} x_\alpha l_\beta + l_4) x_3 + \sum_{s=1}^4 l_s v_3^{(s)} + w_3(x_1, x_2) \right\}, \\ \varphi_{ij} &= (n+1) \left\{ \int_0^{x_3} \varphi_{ij}^* dx_3 + e_{jik} l_k x_3 + \sum_{s=1}^4 l_s \psi_{ij}^{(s)} + \psi'_{ij}(x_1, x_2) \right\}, \end{aligned}$$

where $v_\alpha^{(s)}$, $\psi_{ij}^{(s)}$ ($s=1, 2, 3, 4$) are the solutions of the problems $A^{(s)}$, w_i , ψ'_{ij} are unknown functions and l_s are unknown constants.

Taking into account (19) from equilibrium Eqs. (2) it follows that the functions w_α , $\psi'_{\alpha\beta}$, ψ'_{33} satisfy the equations of the first state of generalized plane strain, also w_3 , $\psi'_{\alpha 3}$, $\psi'_{3\alpha}$ satisfy the equations of the second state of generalized plane strain.

The necessary and sufficient conditions to solve those boundary value problem are satisfied.

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TENSIUNI TERMICE ÎN CILINDRI ELASTICI CU MICROSTRUCTURĂ

(Rezumat)

Se studiază problema tensiunilor termice în cilindri elastici omogeni, izotropi, cu microstructură, folosind unele rezultate stabilite în [1], [2]. Se observă că în acest caz se pot defini două stări de deformare plană generalizată.

RECIPROCITY THEOREM IN THE LINEAR PIEZOELECTRIC MICROPOLAR THERMOELASTICITY THEORY

BY

I. CRĂCIUN

1. Introduction. A dynamical reciprocal theorem of the linear piezoelectric micropolar thermoelasticity theory is derived without using the Laplace transform. The reciprocity relation which we have obtained here, contains only the $\mathbf{u}$, φ , θ , Φ functions and the loads, without the time derivatives of these functions being implied. The theorem covers a nonhomogeneous and anisotropic body subject to arbitrary initial data. From our result the special case of piezoelectric thermoelasticity [2] can be easily obtained.

In what follows we give the index of symbols used in this paper. B — body, ∂B — boundary of B , $\mathcal{E}$ — three-dimensional Euclidian space, $\mathbb{R}^3$ — vector space associated to $\mathcal{E}$, $\times$ — cartesian product, $\mathbf{x}$ — point in $B = B \cup \partial B$, $[0, t_0)$ — time interval, $\mathbf{u}$ — displacement vector, φ — rotation vector, T_0 — reference temperature, T — absolute temperature, $\theta = T - T_0$ — temperature difference, Φ — electric potential, $\mathbf{E}$ — infinitesimal strain tensor, κ — torsion tensor, $\mathbf{g}$ — temperature gradient, $\mathbf{e}$ — electric vector field, $\mathbf{S}$ — stress tensor, $\mathbf{M}$ — couple stress tensor, S — entropy, $\mathbf{d}$ — electric displacement vector, $\mathbf{q}$ — heat flux vector, $\mathbf{A}$ — elasticity tensor, $\mathbf{B}$ — stress-couple stress tensor, $\mathbf{C}$ — tensor similar to $\mathbf{A}$ but related to κ , η — stress-temperature tensor, ξ — couple stress-temperature tensor, $\mathbf{e}$ — piezoelectric-stress tensor, $\mathbf{D}$ — piezoelectric-couple stress tensor, c_ϵ — specific heat, $\mathbf{G}$ — pyroelectric vector, β — dielectric tensor, $\mathbf{K}$ — conductivity tensor, $\mathbf{L}_e$ — Levi-Civita tensor, $\mathbf{X}$ — body force vector, $\mathbf{Y}$ — couple body force vector, r — heat supply, ρ_e — electric charge, ρ — density, $\mathbf{J}$ — micro-inertia tensor, $\mathbf{s}$ — surface traction vector, $\mathbf{m}$ — couple stress vector, q — surface heat charge, σ — surface electric charge, $\mathbf{n}$ — unit outward normal to ∂B , $(\cdot)^T$ — transpose of a tensor, $(\dot{\cdot})$ — time derivative, $(\cdot)_0$ — value of $(\cdot)$ at $t=0$, $(\cdot)$ — inner product, ∇ — gradient, $\nabla \cdot$ — divergence, $\otimes$ — tensorial product, $*$ — convolution, da — element of area, dv — element of volume, $\mathcal{C}_p = \underbrace{\mathbb{R}^3 \otimes \mathbb{R}^3 \otimes \dots \otimes \mathbb{R}^3}_{p \text{ times}}$ — linear space of p -order tensors,

$C^{p,q}(\cdot)$ — class of differentiable functions depending on $\mathbf{x}$ and t .

The body B will be identified with a bounded regular region of $\mathcal{E}$ it occupies in a fixed reference configuration $\mathcal{E}$. In this paper we shall use tensor fields depending on $\mathbf{x}$ and t of the form

$$(1.1) \quad \varphi, \psi, \omega, \dots : \bar{B} \times [0, t_0] \rightarrow \mathcal{T}_p.$$

The convolution $\varphi * \psi$ of the integrable functions φ and ψ is given by

$$(1.2) \quad (\varphi * \psi)(\mathbf{x}, t) = \int_0^t \varphi(\mathbf{x}, \tau) \cdot \psi(\mathbf{x}, t - \tau) d\tau.$$

The properties of convolution are well known (see e. g. [2]—[4]).

2. Basic Relations. By an *admissible process* of the linear theory of piezoelectric micropolar thermoelasticity we mean an ordered array of functions $\pi = [\mathbf{u}, \varphi, \theta, \Phi, \mathbf{E}, \varkappa, \mathbf{g}, \mathbf{e}, \mathbf{S}, \mathbf{M}, S, \mathbf{q}, \mathbf{d}]$ which satisfy the following conditions:

$$(2.1) \quad \begin{aligned} \mathbf{u}, \varphi : \bar{B} \times [0, t_0] &\rightarrow \mathcal{T}_1, \mathbf{u}, \varphi \in C^2(B \times (0, t_0)), \\ \mathbf{u}, \dot{\mathbf{u}}, \ddot{\mathbf{u}}, \nabla \mathbf{u}, \nabla \dot{\mathbf{u}}, \varphi, \dot{\varphi}, \ddot{\varphi}, \nabla \varphi, \nabla \dot{\varphi} &\in C^\circ(\bar{B} \times [0, t_0]), \end{aligned}$$

$$(2.2) \quad \begin{aligned} \theta, \Phi : \bar{B} \times [0, t_0] &\rightarrow \mathbb{R}, \theta, \Phi \in C^{2,1}(B \times (0, t_0)), \\ \theta, \Phi, \nabla \theta, \nabla \Phi, \dot{\theta}, \dot{\Phi}, \nabla \nabla \Phi, \nabla \dot{\Phi} &\in C^\circ(\bar{B} \times [0, t_0]), \end{aligned}$$

$$(2.3) \quad \mathbf{E}, \varkappa : \bar{B} \times [0, t_0] \rightarrow \mathcal{T}_2, \mathbf{E}, \varkappa \in C^\circ(\bar{B} \times [0, t_0]),$$

$$(2.4) \quad \begin{aligned} \mathbf{g}, \mathbf{e} : \bar{B} \times [0, t_0] &\rightarrow \mathcal{T}_1, \mathbf{g}, \mathbf{e} \in C^\circ(\bar{B} \times [0, t_0]), \\ \mathbf{e} \in C^{1,1}(B \times (0, t_0)), \nabla \cdot \mathbf{e}, \dot{\mathbf{e}} &\in C^\circ(\bar{B} \times [0, t_0]), \end{aligned}$$

$$(2.5) \quad \begin{aligned} \mathbf{S}, \mathbf{M} : \bar{B} \times [0, t_0] &\rightarrow \mathcal{T}_2, \mathbf{S}, \mathbf{M} \in C^{1,0}(B \times (0, t_0)), \\ \nabla \cdot \mathbf{S}, \nabla \cdot \mathbf{M}, \mathbf{S}, \mathbf{M} &\in C^\circ(\bar{B} \times [0, t_0]), \end{aligned}$$

$$(2.6) \quad S : \bar{B} \times [0, t_0] \rightarrow \mathbb{R}, S \in C^{0,1}(B \times (0, t_0)), S, \dot{S} \in C^\circ(\bar{B} \times [0, t_0]),$$

$$(2.7) \quad \begin{aligned} \mathbf{q}, \mathbf{d} : \bar{B} \times [0, t_0] &\rightarrow \mathcal{T}_1, \mathbf{q}, \mathbf{d} \in C^{1,0}(B \times (0, t_0)), \\ \mathbf{q}, \mathbf{d}, \nabla \cdot \mathbf{q}, \nabla \cdot \mathbf{d} &\in C^\circ(\bar{B} \times [0, t_0]). \end{aligned}$$

By an *external system of causes* we denote the order array of functions $\mathcal{F} = [\mathbf{X}, \mathbf{Y}, r, \rho_e, \mathbf{s}, \mathbf{m}, q, \sigma]$ in which

$$(2.8) \quad \mathbf{X}, \mathbf{Y} : \bar{B} \times [0, t_0] \rightarrow \mathcal{T}_1, \mathbf{X}, \mathbf{Y} \in C^\circ(\bar{B} \times [0, t_0]),$$

$$(2.9) \quad r, \rho_e : \bar{B} \times [0, t_0] \rightarrow \mathbb{R}, r, \rho_e \in C^\circ(\bar{B} \times [0, t_0]),$$

$$(2.10) \quad \mathbf{s}, \mathbf{m} : \partial B \times [0, t_0] \rightarrow \mathcal{T}_1, \mathbf{s}, \mathbf{m} \in C^\circ(\partial B \times [0, t_0]),$$

$$(2.11) \quad q, \sigma : \partial B \times [0, t_0] \rightarrow \mathbb{R}, q, \sigma \in C^\circ(\partial B \times [0, t_0]),$$

$\mathbf{s}, \mathbf{m}, q$ and σ piecewise regular on ∂B .

By a *piezoelectric micropolar thermoelastic process* corresponding to the external system of causes $\mathcal{F}$ we mean an admissible process π that meets the following equations of the linear piezoelectric micropolar thermoelasticity [1]

$$(2.12) \quad \mathbf{E} = \nabla \mathbf{u} - \varphi \mathbf{L}_e, \quad (2.13) \quad \varkappa = \nabla \varphi, \quad (2.14) \quad \mathbf{g} = \nabla \theta, \quad (2.15) \quad \mathbf{e} = -\nabla \Phi,$$

$$(2.16) \quad \mathbf{S} = \mathbf{A}[\mathbf{E}] + \mathbf{B}[\varkappa] - \theta \boldsymbol{\eta} - \mathbf{e} \boldsymbol{\epsilon}, \quad (2.17) \quad \mathbf{M} = \mathbf{E} \mathbf{B} + \mathbf{C}[\varkappa] - \theta \boldsymbol{\zeta} - \mathbf{e} \mathbf{D},$$

$$(2.18) \quad \mathbf{d} = \mathbf{e}[\mathbf{E}] + \mathbf{D}[\chi] + \theta \mathbf{G} + \beta \mathbf{e}, \quad (2.19) \quad S = \eta \cdot \mathbf{E} + \zeta \cdot \chi + \frac{c_\varepsilon}{T_0} \theta + \mathbf{e} \mathbf{G},$$

$$(2.20) \quad \mathbf{q} = -\mathbf{K} \mathbf{g}, \quad (2.21) \quad \nabla \cdot \mathbf{S} + \mathbf{X} = \rho \ddot{\mathbf{u}}, \quad (2.22) \quad \nabla \cdot \mathbf{M} + \mathbf{L}_c[\mathbf{S}] + \mathbf{Y} = \mathbf{J} \ddot{\boldsymbol{\varphi}},$$

$$(2.23) \quad T_0 \dot{S} + \nabla \cdot \mathbf{q} - r = 0, \quad (2.24) \quad \nabla \cdot \mathbf{d} = \rho_e,$$

where $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{T}_4$, $\mathbf{L}_c, \varepsilon, \mathbf{D} \in \mathcal{T}_3$, $\zeta, \eta, \mathbf{J}, \mathbf{K}, \beta \in \mathcal{T}_2$, $\beta = \beta^T$, $\mathbf{K} = \mathbf{K}^T$ and by $\mathbf{EB}$, for example, we understand the two-order tensors having $E_{kl}B_{klji}$ as components.

At every regular point of $\partial B \times [0, t_0)$, we have

$$(2.25) \quad \mathbf{s} = \mathbf{S}^T \mathbf{n}, \quad (2.26) \quad \mathbf{m} = \mathbf{M}^T \mathbf{n}, \quad (2.27) \quad q = \mathbf{q} \cdot \mathbf{n}, \quad (2.28) \quad \sigma = -\mathbf{d} \cdot \mathbf{n}.$$

The tensors $\mathbf{A}$ and $\mathbf{C}$ satisfy the following symmetries:

$$(2.29) \quad \mathbf{VA} = \mathbf{A}[\mathbf{V}], \quad \mathbf{VC} = \mathbf{C}[\mathbf{V}], \quad (\forall) \mathbf{V} \in \mathcal{T}_2.$$

3. Theorem of Reciprocity. For a given body B and the time interval $[0, t_0)$ let π and $\bar{\pi}$ be piezoelectric micropolar thermoelastic processes corresponding to the external causes $\mathcal{F}$ and $\bar{\mathcal{F}}$, and let π_0 and $\bar{\pi}_0$ be the corresponding initial values of π and $\bar{\pi}$. Then the following integral relation holds

$$(3.1) \quad \begin{aligned} & \int_{\partial B} \left[\mathbf{s} * \bar{\mathbf{u}} + \mathbf{m} * \bar{\boldsymbol{\varphi}} + \frac{1}{T_0} * (q * \bar{\theta}) - \sigma * \bar{\Phi} \right] da + \int_B \left[\mathbf{X} * \bar{\mathbf{u}} + \mathbf{Y} * \bar{\boldsymbol{\varphi}} - \frac{1}{T_0} * (r * \bar{\theta}) - \rho_e * \bar{\Phi} \right] dv + \\ & + \int_B \left[\rho (\dot{\mathbf{u}}_0 \cdot \bar{\mathbf{u}} - \bar{\mathbf{u}}_0 \cdot \dot{\mathbf{u}}) + \mathbf{J} \dot{\boldsymbol{\varphi}}_0 \cdot \bar{\boldsymbol{\varphi}} - \mathbf{J} \bar{\boldsymbol{\varphi}}_0 \cdot \dot{\boldsymbol{\varphi}} - S_0 * \bar{\theta} \right] dv = \\ & = \int_{\partial B} \left[\bar{\mathbf{s}} * \mathbf{u} + \bar{\mathbf{m}} * \boldsymbol{\varphi} + \frac{1}{T_0} * (\bar{q} * \theta) - \bar{\sigma} * \Phi \right] da + \int_B \left[\bar{\mathbf{X}} * \mathbf{u} + \bar{\mathbf{Y}} * \boldsymbol{\varphi} - \frac{1}{T_0} * (\bar{r} * \theta) - \bar{\rho}_e * \Phi \right] dv + \\ & + \int_B \left[\rho (\dot{\bar{\mathbf{u}}}_0 \cdot \mathbf{u} - \mathbf{u}_0 \cdot \dot{\bar{\mathbf{u}}}) + \mathbf{J} \dot{\bar{\boldsymbol{\varphi}}}_0 \cdot \boldsymbol{\varphi} - \mathbf{J} \bar{\boldsymbol{\varphi}}_0 \cdot \dot{\boldsymbol{\varphi}} - \bar{S}_0 * \theta \right] dv. \end{aligned}$$

Proof. Both from the properties of the ∇ -operator and of the convolution, and from (2.12) — (2.14), we obtain

$$(3.2) \quad \nabla \cdot \mathbf{S} * \bar{\mathbf{u}} = \nabla \cdot (\mathbf{S} * \bar{\mathbf{u}}) - \mathbf{S} * \bar{\mathbf{E}} - \mathbf{S} * \bar{\boldsymbol{\varphi}} \mathbf{L}_c, \quad (3.3) \quad \nabla \cdot \mathbf{M} * \bar{\boldsymbol{\varphi}} = \nabla \cdot (\mathbf{M} * \bar{\boldsymbol{\varphi}}) - \mathbf{M} * \bar{\boldsymbol{\chi}},$$

$$(3.4) \quad \nabla \cdot \bar{\mathbf{q}} * \theta = -\nabla \cdot (\mathbf{K} \bar{\mathbf{g}} * \theta) + \mathbf{K} \bar{\mathbf{g}} * \mathbf{g}.$$

From (2.16), (2.17) and the properties of convolution, we have

$$(3.5) \quad \mathbf{S} * \bar{\mathbf{E}} = \mathbf{A}[\mathbf{E}] * \bar{\mathbf{E}} + \mathbf{B}[\chi] * \bar{\mathbf{E}} - \theta * \eta \cdot \bar{\mathbf{E}} - \mathbf{e} * \mathbf{e}[\bar{\mathbf{E}}],$$

$$(3.6) \quad \mathbf{M} * \bar{\boldsymbol{\chi}} = \mathbf{EB} * \bar{\boldsymbol{\chi}} + \mathbf{C}[\chi] * \bar{\boldsymbol{\chi}} - \theta * \zeta \cdot \bar{\boldsymbol{\chi}} - \mathbf{e} * \mathbf{D}[\bar{\boldsymbol{\chi}}].$$

The introduction of (2.19) into (2.23), yields

$$(3.7) \quad \frac{1}{T_0} \nabla \cdot \mathbf{q} + \frac{c_\varepsilon}{T_0} \dot{\theta} + \eta \cdot \dot{\mathbf{E}} + \zeta \cdot \dot{\mathbf{x}} + \mathbf{G} \cdot \dot{\mathbf{e}} = \frac{1}{T_0} r.$$

Taking the convolution of (3.7) with a superposed bar with θ and using (3.4), we get

$$(3.8) \quad \eta \cdot \bar{\mathbf{E}} * \theta + \zeta \cdot \bar{\mathbf{x}} * \theta + \mathbf{G} \cdot \bar{\mathbf{e}} * \theta - \frac{1}{T_0} \nabla \cdot (\mathbf{K} \bar{\mathbf{g}} * \theta) + \frac{1}{T_0} \mathbf{K} \bar{\mathbf{g}} * \mathbf{g} + \frac{c_\varepsilon}{T_0} \bar{\theta} * \theta = \frac{1}{T_0} \bar{r} * \theta.$$

Multiplying (3.8) by 1 in the sense of convolution and using the properties of convolution, we derive

$$(3.9) \quad \eta \cdot \bar{\mathbf{E}} * \theta + \zeta \cdot \bar{\mathbf{x}} * \theta + \mathbf{G} \cdot \bar{\mathbf{e}} * \theta = \frac{1}{T_0} * [\nabla \cdot (\mathbf{K} \bar{\mathbf{g}} * \theta) - \mathbf{K} \bar{\mathbf{g}} * \mathbf{g} + \bar{r} * \theta] - \frac{c}{T_0} \bar{\theta} * \theta + \bar{S}_0 * \theta.$$

Let us multiply now (2.18) by $\bar{\mathbf{e}} = -\nabla \bar{\Phi}$ and use (2.24). We obtain

$$(3.10) \quad \bar{\mathbf{e}} \cdot \mathbf{G} * \theta = \rho_e * \bar{\Phi} - [\nabla \cdot (\mathbf{d} * \bar{\Phi}) + \varepsilon [\mathbf{E}] * \bar{\mathbf{e}} + \mathbf{D}[\mathbf{x}] * \bar{\mathbf{e}} + \beta \mathbf{e} * \bar{\mathbf{e}}].$$

The introduction of (3.10) into (3.9) leads to

$$(3.11) \quad \eta \cdot \bar{\mathbf{E}} * \theta + \zeta \cdot \bar{\mathbf{x}} * \theta = \frac{1}{T_0} * [\nabla \cdot (\mathbf{K} \bar{\mathbf{g}} * \theta) - \mathbf{K} \bar{\mathbf{g}} * \mathbf{g} + \bar{r} * \theta] - \frac{c_\varepsilon}{T_0} \bar{\theta} * \theta + \bar{S}_0 * \theta - \\ - \rho_e * \bar{\Phi} + \nabla \cdot (\mathbf{d} * \bar{\Phi}) + \mathbf{e}[\mathbf{E}] * \bar{\mathbf{e}} + \mathbf{D}[\mathbf{x}] * \bar{\mathbf{e}} + \beta \mathbf{e} * \bar{\mathbf{e}}.$$

By adding (3.5) with (3.6) and by using (3.11), we get

$$(3.12) \quad \mathbf{S} * \bar{\mathbf{E}} + \mathbf{M} * \bar{\mathbf{x}} = \mathbf{A}[\mathbf{E}] * \bar{\mathbf{E}} + \mathbf{B}[\mathbf{x}] * \bar{\mathbf{E}} + \mathbf{E} \mathbf{B} * \bar{\mathbf{x}} + \mathbf{C}[\mathbf{x}] * \bar{\mathbf{x}} - \mathbf{e} * \mathbf{e}[\bar{\mathbf{E}}] - \mathbf{e} * \mathbf{D}[\bar{\mathbf{x}}] - \\ - \frac{1}{T_0} * [\nabla \cdot (\mathbf{K} \bar{\mathbf{g}} * \theta) - \mathbf{K} \bar{\mathbf{g}} * \mathbf{g} + \bar{r} * \theta] + \frac{c_\varepsilon}{T_0} \bar{\theta} * \theta - \bar{S}_0 * \theta + \rho_e * \bar{\Phi} - \\ - \nabla \cdot (\mathbf{d} * \bar{\Phi}) - \mathbf{e}[\bar{\mathbf{E}}] * \bar{\mathbf{e}} - \mathbf{D}[\bar{\mathbf{x}}] * \bar{\mathbf{e}} - \beta \mathbf{e} * \bar{\mathbf{e}}.$$

Taking the convolution both of (2.21) and (2.22) with $\bar{\mathbf{u}}$ und $\bar{\varphi}$, respectively, and using (3.2) and (3.3), we obtain

$$(3.13) \quad \nabla \cdot (\mathbf{S} * \bar{\mathbf{u}}) + \mathbf{X} * \bar{\mathbf{u}} + \rho(\dot{\mathbf{u}}_0 \cdot \bar{\mathbf{u}} - \bar{\mathbf{u}}_0 \cdot \dot{\mathbf{u}}) = \rho \dot{\mathbf{u}} * \bar{\mathbf{u}} + \mathbf{S} * \bar{\mathbf{E}} + \mathbf{S} * \bar{\varphi} \mathbf{L}_e,$$

$$(3.14) \quad \nabla \cdot (\mathbf{M} * \bar{\varphi}) + \mathbf{Y} * \bar{\varphi} + \mathbf{J} \dot{\varphi}_0 \cdot \bar{\varphi} - \mathbf{J} \bar{\varphi}_0 \cdot \dot{\varphi} + \mathbf{L}_e[\mathbf{S}] * \bar{\varphi} = \mathbf{J} \dot{\varphi} * \bar{\varphi} + \mathbf{M} * \bar{\varphi}.$$

Noting that $\mathbf{L}_e[\mathbf{S}] * \bar{\varphi} = \mathbf{S} * \varphi \mathbf{L}_e$, after the addition of (3.13) and (3.14), and the usage of (3.12), we get

$$\begin{aligned}
 & \nabla \cdot (\mathbf{S} * \bar{\mathbf{u}}) + \nabla \cdot (\mathbf{M} * \bar{\varphi}) + \frac{1}{T_0} * \nabla \cdot (\mathbf{K} \bar{\mathbf{g}} * \bar{\theta}) + \nabla \cdot (\mathbf{d} * \bar{\Phi}) + \mathbf{X} * \bar{\mathbf{u}} + \mathbf{Y} * \bar{\varphi} - \\
 & - \rho_e * \bar{\Phi} + \frac{1}{T_0} (\bar{r} * \bar{\theta}) + \rho (\dot{\mathbf{u}}_0 \cdot \bar{\mathbf{u}} - \bar{\mathbf{u}}_0 \cdot \dot{\mathbf{u}}) + \mathbf{J} \dot{\bar{\varphi}}_0 \cdot \bar{\varphi} - \mathbf{J} \bar{\varphi}_0 \cdot \dot{\varphi} + \bar{S}_0 * \bar{\theta} = \\
 (3.15) \quad & = \mathbf{A}[\mathbf{E}] * \bar{\mathbf{E}} + \mathbf{B}[\varkappa] * \bar{\mathbf{E}} + \mathbf{E} \mathbf{B} * \bar{\varkappa} + \mathbf{C}[\varkappa] * \bar{\varkappa} - \mathbf{e} * \mathbf{e}[\bar{\mathbf{E}}] - \mathbf{e}[\mathbf{E}] * \bar{\mathbf{e}} - \mathbf{e} * \mathbf{D}[\bar{\varkappa}] - \\
 & - \mathbf{D}[\varkappa] * \bar{\mathbf{e}} + \rho \dot{\mathbf{u}} * \bar{\mathbf{u}} + \mathbf{J} \dot{\bar{\varphi}} * \bar{\varphi} + \frac{1}{T_0} * (\mathbf{K} \bar{\mathbf{g}} * \bar{\mathbf{g}}) + \frac{c_e}{T_0} \bar{\theta} * \bar{\theta} + \beta \mathbf{e} * \bar{\mathbf{e}}.
 \end{aligned}$$

Taking into account that from the symmetries of $\mathbf{A}$, $\mathbf{C}$, $\mathbf{B}$, $\mathbf{J}$, and $\mathbf{K}$, we have

$$\mathbf{A}[\mathbf{E}] * \bar{\mathbf{E}} = \mathbf{A}[\bar{\mathbf{E}}] * \mathbf{E}, \quad \mathbf{C}[\varkappa] * \bar{\varkappa} = \mathbf{C}[\bar{\varkappa}] * \varkappa, \quad \beta \mathbf{e} * \bar{\mathbf{e}} = \beta \bar{\mathbf{e}} * \mathbf{e},$$

$$\mathbf{K} \bar{\mathbf{g}} * \bar{\mathbf{g}} = \mathbf{K} \mathbf{g} * \bar{\mathbf{g}}, \quad \mathbf{J} \dot{\bar{\varphi}} * \bar{\varphi} = \mathbf{J} \bar{\varphi} * \dot{\varphi},$$

and also that

$$\mathbf{e} * \mathbf{D}[\bar{\varkappa}] + \mathbf{D}[\varkappa] * \bar{\mathbf{e}} = \bar{\mathbf{e}} * \mathbf{D}[\varkappa] + \mathbf{D}[\bar{\varkappa}] * \mathbf{e}, \quad \dot{\mathbf{u}} * \bar{\mathbf{u}} = \bar{\mathbf{u}} * \dot{\mathbf{u}}, \quad \bar{\theta} * \theta = \theta * \bar{\theta}$$

we deduce that, if (3.15) is the equation obtained from (3.15) by replacing the symbols with a bar by the symbols without bars and vice versa, then the right hand sides of (3.15) and (3.15) are identical. Hence, equating their left hand sides, we obtain

$$\begin{aligned}
 & \nabla \cdot (\mathbf{S} * \bar{\mathbf{u}}) + \nabla \cdot (\mathbf{M} * \bar{\varphi}) - \frac{1}{T_0} * \nabla \cdot (\mathbf{K} \bar{\mathbf{g}} * \bar{\theta}) + \nabla \cdot (\mathbf{d} * \bar{\Phi}) + \mathbf{X} * \bar{\mathbf{u}} + \mathbf{Y} * \bar{\varphi} - \\
 & - \frac{1}{T_0} * (r * \bar{\theta}) - \rho_e * \bar{\Phi} + \rho (\dot{\mathbf{u}}_0 \cdot \bar{\mathbf{u}} - \bar{\mathbf{u}}_0 \cdot \dot{\mathbf{u}}) + \mathbf{J} \dot{\bar{\varphi}}_0 \cdot \bar{\varphi} - \mathbf{J} \bar{\varphi}_0 \cdot \dot{\varphi} - S_0 * \bar{\theta} = \\
 (3.16) \quad & = \nabla \cdot (\bar{\mathbf{S}} * \mathbf{u}) + \nabla \cdot (\bar{\mathbf{M}} * \varphi) - \frac{1}{T_0} * \nabla \cdot (\mathbf{K} \bar{\mathbf{g}} * \bar{\theta}) + \nabla \cdot (\bar{\mathbf{d}} * \Phi) + \bar{\mathbf{X}} * \mathbf{u} + \bar{\mathbf{Y}} * \varphi - \\
 & - \frac{1}{T_0} * (\bar{r} * \theta) - \bar{\rho}_e * \Phi + \bar{\rho} (\dot{\mathbf{u}}_0 \cdot \mathbf{u} - \mathbf{u}_0 \cdot \dot{\mathbf{u}}) + \bar{\mathbf{J}} \dot{\varphi}_0 \cdot \varphi - \bar{\mathbf{J}} \varphi_0 \cdot \dot{\varphi} - \bar{S}_0 * \theta.
 \end{aligned}$$

From the divergence theorem and (2.25) – (2.28), we derive

$$(3.17) \quad \int_B \nabla \cdot (\mathbf{S} * \bar{\mathbf{u}}) dv = \int_{\partial B} \mathbf{s} * \bar{\mathbf{u}} da, \quad (3.18) \quad \int_B \nabla \cdot (\mathbf{M} * \bar{\varphi}) dv = \int_{\partial B} \mathbf{m} * \bar{\varphi} da,$$

$$(3.19) \quad \int_B \nabla \cdot (\mathbf{K} \bar{\mathbf{g}} * \bar{\theta}) dv = - \int_{\partial B} q * \bar{\theta} da, \quad (3.20) \quad \int_B \nabla \cdot (\mathbf{d} * \bar{\Phi}) dv = - \int_{\partial B} \sigma * \bar{\Phi} da.$$

Finally, if we integrate (3.16) over B and use (3.17) – (3.20), we arrive at (3.1).

As special case we can give the reciprocal theorem for the piezoelectric thermoelasticity [2] by neglecting in (3.1) all the terms in which only the rotation appears. In this way, we obtain

$$\begin{aligned} & \int_{\partial B} \left[\mathbf{s}^* \bar{\mathbf{u}} + \frac{1}{T_0} * (\bar{q}^* \bar{\theta}) - \bar{\sigma}^* \bar{\Phi} \right] d\alpha + \int_B \left[\mathbf{X}^* \bar{\mathbf{u}} - \frac{1}{T_0} * (r^* \bar{\theta}) - \bar{\rho}_e^* \bar{\Phi} \right] dv + \\ & + \int_B [\rho (\dot{\mathbf{u}}_0 \cdot \bar{\mathbf{u}} - \bar{\mathbf{u}}_0 \cdot \dot{\mathbf{u}}) + \mathbf{J} \dot{\varphi}_0 \cdot \bar{\varphi} - \mathbf{J} \bar{\varphi}_0 \cdot \dot{\varphi} - S_0^* \bar{\theta}] dv = \\ & = \int_{\partial B} \left[\dot{\mathbf{s}}^* \mathbf{u} + \frac{1}{T_0} * (\dot{\bar{q}}^* \bar{\theta}) - \bar{\sigma}^* \Phi \right] d\alpha + \int_B \left[\dot{\mathbf{X}}^* \mathbf{u} - \frac{1}{T_0} * (r^* \bar{\theta}) - \bar{\rho}_e^* \Phi \right] dv + \\ & + \int_B [\rho (\dot{\mathbf{u}}_0 \cdot \mathbf{u} - \mathbf{u}_0 \cdot \dot{\mathbf{u}}) + \mathbf{J} \dot{\varphi}_0 \cdot \varphi - \mathbf{J} \varphi_0 \cdot \dot{\varphi} - \bar{S}_0^* \bar{\theta}] dv. \end{aligned}$$

Other special cases may be obtained if we suppose that the initial data are homogeneous, or the thermal effects are absent, etc.

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TEOREMĂ DE RECIPROCIȚATE ÎN TEORIA LINIARĂ A TERMOELASTICITĂȚII MICROPOLARE PIEZOELECTRICE

(Rezumat)

Se deduce o teoremă de reciprocitate pentru teoria liniară a termoelasticității micropolare piezoelectrice fără a se folosi transformata Laplace. Spre deosebire de [1], relația de reciprocitate obținută aici nu conține derivatele în raport cu timpul a deplasărilor, rotațiilor, variației de temperatură și a potențialului electric.

STABILITY OF THE SOLUTIONS IN LINEAR MICROPOLAR VISCOELASTICITY (II)

BY

D. MANOLE

1. Introduction. Sufficent stability conditions for the solutions of linear dynamic micropolar viscoelasticity was investigated in [2]. Conditions are established ensuring the continuous dependence on the initial data of the equilibrium solution. The method of proof depends upon logarithmic convexity arguments.

2. Preliminary Considerations. Throughout this paper we employ a rectangular coordinate system x_k and the indicial notation. We consider a regular (in the sense of Kellogg) region Ω of space occupied by a micropolar viscoelastic material, whose boundary $\partial\Omega$ in the time interval $[0, T]$, $0 < T < \infty$. The basic equations of the linear dynamic theory of micropolar viscoelasticity are

1) the equation of motion

$$(2.1) \quad t_{ji,j} + F_i = \rho(x) \ddot{u}_i, \quad m_{ji,i} + e_{ijk} t_{jk} + M_i = I_{ij}(x) \ddot{\omega}_j;$$

2) the constitutive equations

$$(2.2) \quad \begin{aligned} t_{ij}(x, t) &= A_{ijk}(x, t) \gamma_{kl}(x, t) + B_{ijk}(x, t) x_{kl}(x, t) + \\ &+ \int_0^t \dot{A}_{ijk}(x, t, s) \gamma_{kl}(x, s) ds + \int_0^t \dot{B}_{ijk}(x, t, s) x_{kl}(x, s) ds, \\ m_{ij}(x, t) &= B_{klij}(x, t) \gamma_{kl}(x, t) + C_{ijk}(x, t) x_{kl}(x, t) + \\ &+ \int_0^t \dot{B}_{klij}(x, t, s) \gamma_{kl}(x, s) ds + \int_0^t \dot{C}_{ijk}(x, t, s) x_{kl}(x, s) ds; \end{aligned}$$

3) the geometrical equations

$$(2.3) \quad \gamma_{ij} = u_{i,j} - e_{kij} \omega_k, \quad x_{ij} = \omega_{j,i}.$$

In these relations we have used the following notations: t_{ij} — components of the stress tensor; m_{ij} — components of the couple stress tensor; F_i — components of the body force vector; M_i — components of the body couple vector; γ_{ij} , κ_{ij} — components of the micropolar strain tensor; u_i — components of the displacement vector; ω_i — components of the microrotation vector; e_{ijk} — alternating symbol; ρ , I_{ij} , A_{ijkl} , B_{ijkl} , C_{ijkl} — characteristics of the material; the comma denotes partial derivation with respect to the space variables x_k and a dot denotes partial derivation with respect to the time t .

To the system of field equations we adjoin the initial conditions

$$(2.4) \quad \begin{aligned} u_i(x, 0) &= a_i(x), & \omega_i(x, 0) &= b_i(x), & x &\in \Omega, \\ \dot{u}_i(x, 0) &= c_i(x), & \dot{\omega}_i(x, 0) &= d_i(x), & x &\in \Omega, \end{aligned}$$

and the homogeneous boundary conditions

$$(2.5) \quad \begin{aligned} u_i(x, t) &= 0, \text{ on } \partial\Omega_u \times [0, T], & \omega_i(x, t) &= 0, \text{ on } \partial\Omega_\omega \times [0, T], \\ t_i(x, t) &= t_{ji}n_j = 0 \text{ on } \partial\Omega_t \times [0, T], & m_i(x, t) &= m_{ji}n_j = 0, \text{ on } \partial\Omega_m \times [0, T], \end{aligned}$$

where a_i , b_i , c_i , d_i are prescribed functions, n_i are components of the unit outward normal to $\partial\Omega$, and $\partial\Omega_u$, $\partial\Omega_\omega$, $\partial\Omega_t$, $\partial\Omega_m$ denote subset of such that

$$\partial\Omega = \partial\Omega_u \cup \partial\Omega_t = \partial\Omega_\omega \cup \partial\Omega_m, \quad \partial\Omega_u \cap \partial\Omega_t = \partial\Omega_\omega \cap \partial\Omega_m = \emptyset.$$

We introduce the notations

$$(2.6) \quad \mathcal{A}(\gamma_{ij}, \kappa_{ij}) = A_{ijkl}\gamma_{ij}\gamma_{kl} + 2B_{ijkl}\gamma_{ij}\kappa_{kl} + C_{ijkl}\kappa_{ij}\kappa_{kl},$$

$$(2.7) \quad E(t) = \frac{1}{2} \int_{\Omega} (\rho \dot{u}_i^2 + I_{ij} \dot{\omega}_i \dot{\omega}_j + A_{ijkl} \gamma_{ij} \gamma_{kl} + 2B_{ijkl} \gamma_{ij} \kappa_{kl} + C_{ijkl} \kappa_{ij} \kappa_{kl}) d\Omega,$$

$$(2.8) \quad t_{ij}^{(V)} = \int_0^t (A_{ijkl} \dot{\gamma}_{kl} + B_{ijkl} \dot{\kappa}_{kl}) ds, \quad m_{ij}^{(V)} = \int_0^t (B_{ijkl} \dot{\gamma}_{kl} + C_{ijkl} \dot{\kappa}_{kl}) ds,$$

$$(2.9) \quad P(t) = \int_{\Omega} (F_i \dot{u}_i + M_i \dot{\omega}_i) d\Omega.$$

The characteristics of the material obey

$$(2.10) \quad A_{ijkl} = A_{klij}, \quad C_{ijkl} = C_{klij};$$

$$(2.11) \quad \begin{aligned} \rho(x) &\geq \rho_0 > 0; \\ I_{ij}(x) \xi_i \xi_j &\geq \gamma \xi_k \xi_k, & I_{ij}(x) &= I_{ji}(x), \quad \forall \xi(\xi_k), \quad \gamma > 0. \end{aligned}$$

We suppose

$$(2.12) \quad \mathcal{A}(\gamma_{ij}, \kappa_{ij}) \geq c \sum_{i,j=1}^3 (\gamma_{ij}^2 + \kappa_{ij}^2), \quad c > 0, \quad x \in \Omega,$$

$$(2.13) \quad - \int_{\Omega} (\dot{A}_{ijk} \gamma_{ij} \gamma_{kl} + 2\dot{B}_{ijk} \gamma_{ij} \kappa_{kl} + \dot{C}_{ijk} \kappa_{ij} \kappa_{kl}) d\Omega > \delta \sum_{i,j=1}^3 (\gamma_{ij}^2 + \kappa_{ij}^2), \quad \delta > 0,$$

$$(2.14) \quad \int_{\Omega} (l_{ij}^{(V)} \dot{\gamma}_{ij} + m_{ij}^{(V)} \dot{\kappa}_{ij}) d\Omega \geq 0, \quad t \in [0, T].$$

3. Stability Analysis. The null solution is stable under perturbation u_i, ω_i satisfying (2.1)–(2.5) if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that [1]

$$(3.1) \quad \left[\int_{\Omega(0)} (\rho u_i u_i + I_{ij} \omega_i \omega_j) d\Omega + Q \right] < \delta,$$

implies

$$(3.2) \quad \sup_{0 \leq t < T} \left[\int_{\Omega(t)} (\rho u_i u_i + I_{ij} \omega_i \omega_j) d\Omega + Q \right] < \varepsilon,$$

where $\Omega(t)$ denotes integration over the volume of the body at time t , while Q is an appropriately chosen positive functional of the initial data which tends to zero as the initial data tends to zero. Its precise form will be specified later. We say that a solution is unstable when it is not stable.

Theorem 3.1. *For the system of Eqs. (2.1)–(2.5) the null solution is stable provided the perturbations satisfy the uniform boundedness condition*

$$(3.3) \quad \int_{\Omega(t)} (\rho u_i u_i + I_{ij} \omega_i \omega_j) d\Omega < M^2$$

for some positive bounded constant M .

Proof. Consider the function $G(t)$ defined by

$$(3.4) \quad G(t) = \ln[F(t) + Q] + t^2,$$

where

$$(3.5) \quad F(t) = \int_{\Omega(t)} (\rho u_i u_i + I_{ij} \omega_i \omega_j) d\Omega.$$

We have

$$(3.6) \quad (F+Q)^2 \ddot{G} \equiv (F+Q) \ddot{F} - (\dot{F})^2 + 2(F+Q)^2 \geq 0, \quad 0 \leq t \leq T.$$

Indeed from (3.5) and (2.11) we obtain

$$(3.7) \quad \dot{F} = 2 \int_{\Omega(t)} (\rho u_i \dot{u}_i + I_{ij} \omega_i \dot{\omega}_j) d\Omega$$

and

$$(3.8) \quad \ddot{F} = 2 \int_{\Omega(t)} (\rho \dot{u}_i \dot{u}_i + I_{ij} \dot{\omega}_i \dot{\omega}_j + \rho u_i \ddot{u}_i + I_{ij} \omega_i \ddot{\omega}_j) d\Omega.$$

Applying the divergence theorem and taking into account (2.1), (2.5) from (3.8) we have

$$(3.9) \quad \ddot{F} = 2 \int_{\Omega(t)} [\rho \ddot{u}_i \dot{u}_i + I_{ij} \ddot{\omega}_i \dot{\omega}_j - (t_{ij} \dot{\gamma}_{ij} + m_{ij} \dot{\chi}_{ij})] d\Omega.$$

Multiplying (2.1)₁ by $\dot{U}_i(x, t)$ and (2.1)₂ by $\dot{\omega}_i(x, t)$, adding, integrating over $\Omega(t)$ applying the divergence theorem and taking into account (2.5) it results

$$(3.10) \quad \int_{\Omega(t)} (\rho \ddot{u}_i \dot{u}_i + I_{ij} \ddot{\omega}_i \dot{\omega}_j + t_{ij} \dot{\gamma}_{ij} + m_{ij} \dot{\chi}_{ij}) d\Omega = \int_{\Omega(t)} [F_i \dot{u}_i + M_i \dot{\omega}_i] d\Omega.$$

From (3.10) on the basis of (2.2) we obtain

$$(3.11) \quad \int_{\Omega(t)} (\rho \ddot{u}_i \dot{u}_i + I_{ij} \ddot{\omega}_i \dot{\omega}_j + A_{ijk} l \dot{\gamma}_{kl} \dot{\gamma}_{ij} + B_{ijk} l \dot{\chi}_{kl} \dot{\gamma}_{ij} + B_{klij} \dot{\gamma}_{kl} \dot{\chi}_{ij} + C_{ijk} l \dot{\chi}_{kl} \dot{\chi}_{ij}) d\Omega + \\ + \int_{\Omega(t)} \int_0^t (\dot{A}_{ijk} l \dot{\gamma}_{kl} \dot{\gamma}_{ij} + \dot{B}_{ijk} l \dot{\chi}_{kl} \dot{\gamma}_{ij} + \dot{B}_{klij} \dot{\gamma}_{kl} \dot{\chi}_{ij} + \dot{C}_{ijk} l \dot{\chi}_{kl} \dot{\chi}_{ij}) ds d\Omega = \\ = \int_{\Omega(t)} (F_i \dot{u}_i + M_i \dot{\omega}_i) d\Omega.$$

Using the relations (2.7), (2.8), (2.9) from (3.11) we have

$$(3.12) \quad \dot{E}(t) = \frac{1}{2} \int_{\Omega(t)} (\dot{A}_{ijk} l \dot{\gamma}_{kl} \dot{\gamma}_{ij} + 2 \dot{B}_{ijk} l \dot{\gamma}_{kl} \dot{\gamma}_{ij} \dot{\chi}_{kl} + \dot{C}_{ijk} l \dot{\chi}_{kl} \dot{\chi}_{ij}) d\Omega - \\ - \int_{\Omega(t)} (t_{ij}^{(V)} \dot{\gamma}_{ij} + m_{ij}^{(V)} \dot{\chi}_{ij}) d\Omega + P(t).$$

In equilibrium case $F_i(x, t) = 0$, $M_i(x, t) = 0$. In this case $P(t) = 0$ and by (3.12), (2.13), (2.14) it results

$$(3.13) \quad E(t) \leq E(0), \quad \text{for } t > 0.$$

From (2.7), (2.13), (2.14), we may use Schwarz's inequality to obtain

$$(3.14) \quad (F+Q)\ddot{F} - (\dot{F})^2 \geq -2Q(F+Q) \geq -2(F+Q)^2,$$

provided Q is chosen to satisfy $Q \geq 2E(0)$.

Thus (3.6) is established. From (3.6) it results the convexity of $G(t)$ on $0, T$. From the convexity of $G(t)$ it immediately follows that

$$(3.15) \quad G(t) = G \left[\frac{t}{T} T + \left(1 - \frac{t}{T} \right) 0 \right] \leq \frac{t}{T} G(T) + \left(1 - \frac{t}{T} \right) G(0), \quad 0 \leq t \leq T,$$

i.e.

$$(3.16) \quad F(t) + Q \leq e^{t(T-t)} [F(T) + Q]^{t/T} [F(0) + Q]^{1-t/T}, \quad 0 \leq t \leq T.$$

Since all terms on the right of (3.16) remain bounded it follows that for $0 \leq t < T$ arbitrarily small values of $F(0) + Q$ imply arbitrarily small values of $F(t) + Q$. This concludes the proof of the theorem.

Theorem 3.2. *If*

$$(3.17) \quad F_t = -c_1(x) \dot{u}_t, \quad M_t = -c_2(x) \dot{\omega}_t, \quad c_1(x) \geq 0, \quad c_2(x) \geq 0,$$

we have (3.13).

Proof. From (3.17) on the basis of (2.9) we obtain

$$(3.18) \quad P(t) \leq 0.$$

Using the relations (3.12), (3.18), (2.13), (2.14) it results

$$(3.19) \quad \dot{E}(t) \leq 0, \quad \text{for } t > 0,$$

i.e. (3.13). The relation (3.13) [2] takes place for $\partial\Omega_t = \partial\Omega_m = \emptyset$.

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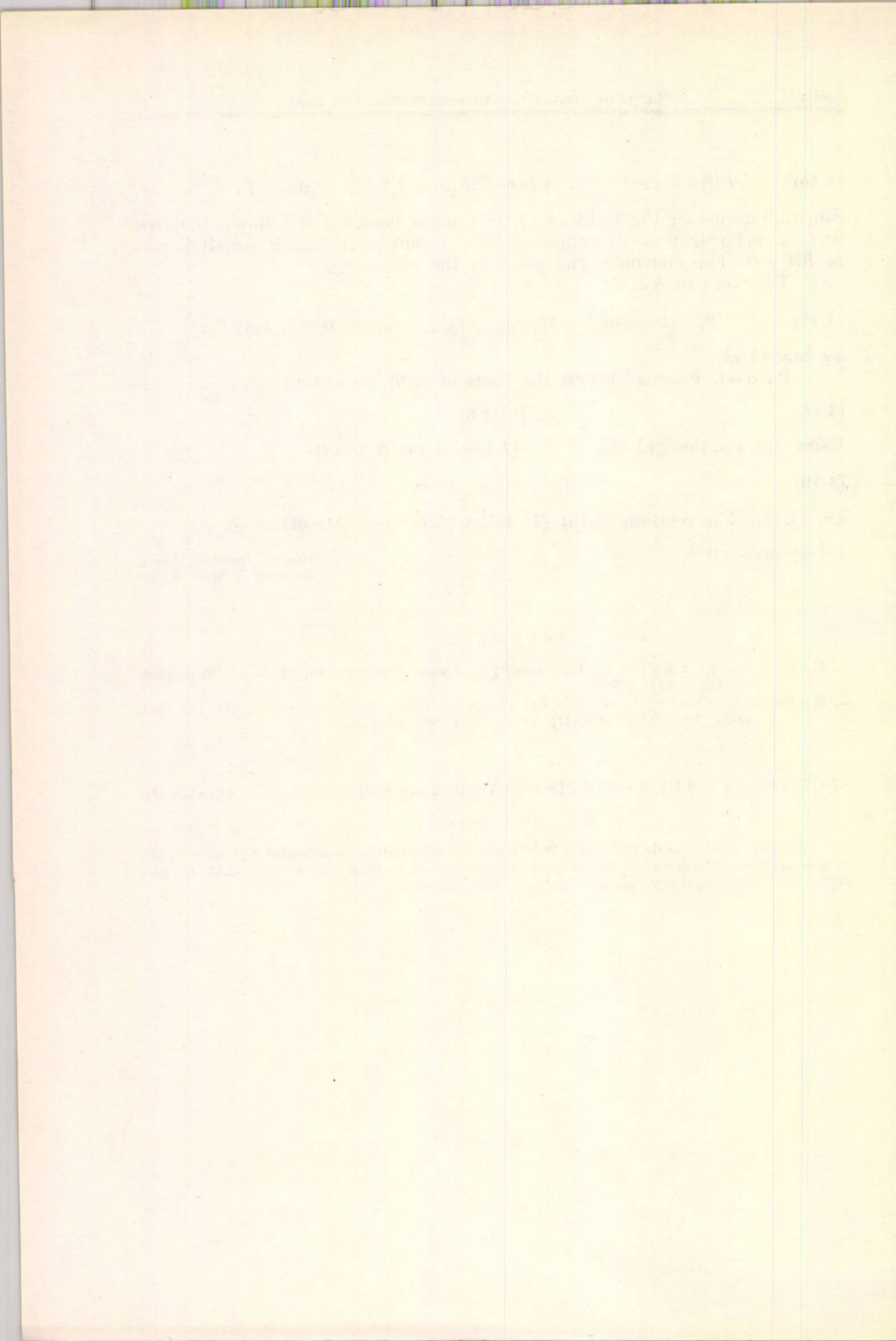
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STABILITATEA SOLUȚIILOR ÎN VÎSCOELASTICITATEA MICROPOLARĂ LINIARĂ (II)

(Rezumat)

Condiții suficiente de stabilitate în viscoelasticitatea micropolară liniară sînt date în [2]. Aici se arată că soluția nulă a teoriei viscoelasticității liniare micropolare este stabilă folosind faptul că o anumită funcție aleasă convenabil este convexă.



GAUGE THEORY ON $\mathbb{R} \times S^3$ TOPOLOGY

BY

GH. ZET

1. Introduction. The field theory on $\mathbb{R} \times S^3$ topology has been considered recently by different authors [1]—[7]. The explanation is that the surface of a four-dimensional sphere S^3 is a space with considerably more symmetry. It is convenient to leave the temporal dimension continuous and flat, so the theory can be canonically quantized.

In this paper we give a model for gauge theories on $\mathbb{R} \times S^3$ topology. First, the geometry of $\mathbb{R} \times S^3$ space is reviewed, and then the gauge theory over a compact Lie group G is described using $\mathbb{R} \times S^3$ as background space. The $U(1)$ and $SU(2)$ gauge theories on $\mathbb{R} \times S^3$ topology are considered as particular examples and a comparison with other results is given. Finally, some observations about the supersymmetry and gravity on $\mathbb{R} \times S^3$ are made.

2. The Geometry of $\mathbb{R} \times S^3$. The sphere S^3 can be parametrized [8], [9] in terms of either cartesian coordinates x_m ($m=1, 2, 3, 4$), complex variables (u, v) or angles (θ, α, β) . If the radius of S^3 is equal to ρ , then

$$(1) \quad x_1^2 + x_2^2 + x_3^2 + x_4^2 = \rho^2.$$

We will set ρ equal to unity ($\rho=1$) wherever possible. The inverse radius ρ^{-1} can be considered as a mass scale in this space. The connection between different variables is of the form

$$(2) \quad \begin{aligned} u &= x_1 + ix_2 = \cos \theta e^{i\alpha}, \\ v &= x_3 + ix_4 = \cos \theta e^{i\beta}, \end{aligned}$$

with the volume element

$$(3) \quad d\Omega = \frac{1}{2\pi^2} \sin \theta \cos \theta d\theta d\alpha d\beta.$$

The manifold has an $O(4)$ symmetry, whose Lie algebra is $SU(2) \times SU(2)$. S^3 is itself the manifold of the group $SU(2)$, and the two subgroups correspond to left and right group multiplications acting on the manifold. As a consequence, there are two independent frames $e_i^{(1)}$ and $e_i^{(2)}$ ($i=1, 2, 3$) possible on S^3 , each of which is smooth everywhere and can be used to define the spatial derivatives. The particular choice of one or other of these frames has no physical consequences.

Throughout this paper we choose the frame $e_i^{(1)}$ and the corresponding derivatives will be denoted by ∂_i ($i=1, 2, 3$):

$$\begin{aligned}
 \partial_1 &= i \left(u \frac{\partial}{\partial v^*} - v \frac{\partial}{\partial u^*} + v^* \frac{\partial}{\partial u} - u^* \frac{\partial}{\partial v} \right) = \\
 &= \sin(\alpha + \beta) \frac{\partial}{\partial \theta} + \cos(\alpha + \beta) \left[\operatorname{tg} \theta \frac{\partial}{\partial \alpha} - \operatorname{ctg} \theta \frac{\partial}{\partial \beta} \right], \\
 \partial_2 &= u \frac{\partial}{\partial v^*} - v \frac{\partial}{\partial u^*} - v^* \frac{\partial}{\partial u} + u^* \frac{\partial}{\partial v} = \\
 &= \cos(\alpha + \beta) \frac{\partial}{\partial \theta} + \sin(\alpha + \beta) \left[\operatorname{tg} \theta \frac{\partial}{\partial \alpha} - \operatorname{ctg} \theta \frac{\partial}{\partial \beta} \right], \\
 \partial_3 &= i \left(u^* \frac{\partial}{\partial u^*} - u \frac{\partial}{\partial u} + v^* \frac{\partial}{\partial v^*} - v \frac{\partial}{\partial v} \right) = -\frac{\partial}{\partial \alpha} - \frac{\partial}{\partial \beta}.
 \end{aligned}
 \tag{4}$$

Note that these derivatives do not commute

$$[\partial_i, \partial_j] = 2\varepsilon_{ijk} \partial_k, \tag{5}$$

where ε_{ijk} is the antisymmetric tensor with $\varepsilon_{123}=1$. If we consider the derivatives $\partial_\mu = (\partial_0 = \partial/\partial t, \partial_i)$, then we have

$$[\partial_\mu, \partial_\nu] = 2\varepsilon_{0\mu\nu\gamma} \partial_\gamma. \tag{5'}$$

In Carmeli and Malin's works [1]–[6] one uses the angular momentum components L_i ($i=1, 2, 3$) as derivatives in S^3 . They satisfy the commutation relations

$$[L_i, L_j] = -\varepsilon_{ijk} L_k, \tag{6}$$

which are equivalent up to a factor with (5).

3. Gauge Theory Over a Lie Group on $\mathbb{R} \times S^3$ Topology. Let G be a Lie group whose generators are X_A , $A=1, 2, \dots, n$. The structure equations of G are

$$[X_A, X_B] = f_{AB}^C X_C, \tag{7}$$

where $f_{AB}^C = -f_{BA}^C$ are the structure constants of G . Let us suppose that G is a symmetry group for a Lagrangean associated to some fields defined over $\mathbb{R} \times S^3$. Then, we impose the condition of local symmetry under the group G . We construct a principal bundle $P(G, \mathbb{R} \times S^3)$ with structure group G and base manifold $\mathbb{R} \times S^3$. Introduce a connection in P by defining the covariant derivatives

$$D_\mu = \partial_\mu + ig h_\mu^A X_A, \tag{8}$$

where $h_\mu^A(t, \theta, \alpha, \beta) = h^A(x^\nu)$ denotes the gauge fields (or connection coefficients) associated to the generators X_A and g is the gauge coupling constant. The derivatives ∂_μ are taken as (∂_0, ∂_i) with ∂_i satisfying (5'), or (∂_0, L_i) with L_i satisfying (6). We suppose that the fields h_μ^A belongs to the adjoint representation of G , i.e.

$$(9) \quad X_A(h^B) = f_{CA}{}^B h^C$$

and transforms as a vector (with respect to the index μ) under general coordinate transformations. Then we have

$$[D_\mu, X_A] = 0.$$

Under a gauge transformation generated by $\epsilon^A X_A$ we have

$$(10) \quad \delta_\epsilon D_\mu = [D_\mu, \epsilon^A X_A] = (D_\mu \epsilon^A) X_A.$$

The gauge fields h_μ^A transform then under the law

$$(11) \quad \delta_\epsilon h_\mu^A = D_\mu \epsilon^A = \partial_\mu \epsilon^A + i g f_{BC}^A h_\mu^B \epsilon^C.$$

The curvature $\mathcal{Q}$ is a horizontal two-form with values in the Lie algebra of G , whose components are given by

$$(12) \quad \mathcal{Q}(D_\mu, D_\nu) = \frac{1}{ig} [D_\mu, D_\nu] - \frac{1}{ig} D_{[\partial_\mu, \partial_\nu]}.$$

Using (8) and (5'), the components of the curvature two-form (12) become

$$(13) \quad R_{\mu\nu}^A = \partial_\mu h_\nu^A - \partial_\nu h_\mu^A + i g f_{BC}^A h_\nu^B h_\mu^C - 2 \epsilon_{0\mu\nu\gamma} h_\gamma^A,$$

where in the last term the sum over the $\gamma=0, 1, 2, 3$ index is taken. This term appears because the derivatives ∂_μ do not commute. If these derivatives would be commutative, i.e. $[\partial_\mu, \partial_\nu]=0$, then the last term in (12) is null. This is the case of the gauge theories over the Minkowski space.

If we define $h_\mu = h_\mu^A X_A$ and $F_{\mu\nu} = R_{\mu\nu}^A X_A$, then we can write

$$(14) \quad F_{\mu\nu} = \partial_\mu h_\nu - \partial_\nu h_\mu + i g [h_\mu, h_\nu] - 2 \epsilon_{0\mu\nu\gamma} h_\gamma.$$

The curvature transforms under the gauge group G as follows

$$(15) \quad \delta_\epsilon R_{\mu\nu}^A = f_{BC}^A \epsilon^B R_{\mu\nu}^C.$$

In the next sections we will apply these results to the cases when the symmetry group G is $U(1)$ and $SU(2)$.

4. $U(1)$ Gauge Theory. Electrodynamics. In the symmetry group G is $U(1)$, then we have $A=1$, $f_{BC}^A=0$ ($U(1)$ is abelian) and denoting $h_\mu^1 \equiv A_\mu$, the curvature components (13) become

$$(16) \quad F_{\mu\nu} \equiv R_{\mu\nu}^1 = \partial_\mu A_\nu - \partial_\nu A_\mu - 2 \epsilon_{0\mu\nu\gamma} A_\gamma.$$

This result is in concordance with that obtained in [7] but differs from Carmeli and Malin's result [4] by the last term on the right-hand side. Namely, if we use the derivatives $L_\mu = (\partial_0, L_i)$ as in [4], then

$$(17) \quad F_{\mu\nu} = R_{\mu\nu}^1 = L_\mu A_\nu - L_\nu A_\mu + \epsilon_{0\mu\nu\gamma} A_\gamma.$$

The magnetic and electric fields are related to the tensor field $F_{\mu\nu}$ by

$$(18) \quad H_i = F_{i0} = F^{0i} = \partial_i A_0 - \partial_0 A_i,$$

$$(19) \quad E_i = \frac{1}{2} \epsilon_{ijk} F_{jk} = \epsilon_{ijk} \partial_j A_k - 2 A_i,$$

where A_0 is a scalar function of t , θ , α and β variables. This result coincides also with that given in [7] but differs by the term $-2A_i$ from that of [4]. In fact, with the Carmeli and Malin's notations, the last term in (19) is equal to $+A_i$.

The Lagrangean of the electromagnetic field A_μ (gauge field) has the form

$$(20) \quad \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}.$$

The equations for electrodynamics can be written then, considering, for example, the frame $e_i^{(1)}$. In this case, for a vector $\vec{V}$ on S^3 with the components V_i we can write

$$(21) \quad (\text{rot } \vec{V})_i = (\nabla \times \vec{V})_i = \varepsilon_{ijk} \partial_j V_k - 2V_i.$$

Then, Eqs. (18) and (19) become

$$(22) \quad H_i = \partial_i A_0 - \partial_0 A_i,$$

and respectively

$$(23) \quad E_i = (\nabla \times \vec{A})_i.$$

Methods for obtaining solutions of the electrodynamics equations were given in [4], but they must be rewritten using the new expression for E_i . The bosonic and fermionic fields in $\mathbb{R} \times S^3$ are obtained in [7].

5. $SU(2)$ Gauge Theory. Let us suppose now that G is the group $SU(2)$; then $A=1, 2, 3=i$, and the generators of $SU(2)$ are $X_A=X_i$. The gauge fields are then of the form $h_\mu^i = h_\mu^i(t, \theta, \alpha, \beta)$, $i=1, 2, 3$ and $\mu=0, 1, 2, 3$. The curvatures components associated to these fields are given by Eq. (13), which in $SU(2)$ gauge becomes

$$(24) \quad R_{\mu\nu}^i = \partial_\mu h_\nu^i - \partial_\nu h_\mu^i + ig f_{jk}^i h_\nu^j h_\mu^k - 2\varepsilon_{0\mu\nu\gamma} h_\gamma^i.$$

Here, f_{jk}^i are the structure constants of $SU(2)$ group. If we define the matrices

$$(25) \quad B_\mu = h^i X_i, \quad F_{\mu\nu} = R_{\mu\nu}^i X_i,$$

then the curvature (24) can be written now under the form

$$(26) \quad F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + ig [B_\mu, B_\nu] - 2\varepsilon_{0\mu\nu\gamma} B_\gamma.$$

The gauge fields transform under the gauge group as

$$(27) \quad \delta_\varepsilon h_\mu^i = \partial_\mu \varepsilon^i + ig f_{jk}^i h_\mu^j \varepsilon^k.$$

One observes that the last term in (26) do not appears in [5], and it is determined by the non-commutativity of the derivatives ∂_i .

The Lagrangean of the gauge field and a spin $-\frac{1}{2}$ field ψ is given by

$$(28) \quad \mathcal{L} = -\frac{1}{8} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) - \bar{\psi} \gamma^\mu (\partial_\mu - ie B_\mu) \psi - I \bar{\psi} \psi,$$

where I is the rest momentum of inertia of the particle [5]. The currents and field equations are given in [5], but with the above observations.

The global supersymmetry $N=1$ was also implemented on the space $\mathbb{R} \times S^3$ [7]. For cosmological applications these results must be generalized to the case of a time-dependent radius of curvature of S^3 , and also to investigate how theories of gravity and supergravity look when the background is $\mathbb{R} \times S^3$. This remains an open question for new researches, and it is suggested [7] that this may however lead to difficulties, since it was shown that the supersymmetry in $\mathbb{R} \times S^3$ is not consistent with conformal and general coordinate invariance.

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TEORIE DE ETALONARE PE TOPOLOGIA DIN $\mathbb{R} \times S^3$

(Rezumat)

Se expune un model pentru teorii de etalonare pe topologia din $\mathbb{R} \times S^3$, din care teoriile de etalonare $U(1)$ și $SU(2)$ pe această topologie se obțin ca exemple particulare. În final se prezintă unele observații privind suprasimetria și gravitatea pe $\mathbb{R} \times S^3$.

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THE DENSITY OF PHONONIC STATES IN THE SEMICONDUCTORS WITH POSITION DEPENDENT BAND STRUCTURE

BY

MARIANA ISTRATE, CARMEN CARASEVICI and SERGIU ISTRATE

1. Introduction. Position-dependent band structure occur in various devices which utilize a varying composition of components, such as graded band gap, binary mixtures, graded heterojunctions, etc. The devices employing materials which are intrinsically homogeneous, but which are rendered inhomogeneous by the application of stress, strong temperature gradient, or by nonuniform heavy doping has also a position dependent band structure. In the semiconductors with nonuniform band structure, the bottom of the conduction band E_c , the top of the valence band E_v , the band gap E_g , the density of electron states g and the electron affinity χ are position-dependent.

Bardeen and Shockley [1] first showed that the concept of a variable band gap is legitimate for a semiconductor with a varying elastic strain. S. K. Chattopadhye and V. Mathur [2] have considered the carrier distribution in graded band gap semiconductor under asymmetric band edge gradients. P. R. Em t a g e [3] has related the photovoltaic effect to the variable band gap present in such materials as graded heterojunctions and alloys. T. Gora and F. Williams [4] have approached the theory of electronic states and transport in graded mixed semiconductors. A. Marshak and K. M. van Vliet have related the electrical current [5], the carrier densities [6], the law of the junction [7], the Shockley-like equation [8] and the effective density of states [9] in the semiconductors with nonuniform band structure.

In previous papers [10]—[13] we have presented a theoretical study of the relaxation time, the drag effect, the density of current in photoelectrolysis and the density of electronic states in semiconductors with position-dependent band structure.

2. Density of Phonon States. In the semiconductors with position-dependent band structure the lattice has a nonhomogeneous structure. The phonon frequency and the phonon energy is dependent by the volume and temperature [14], [15] and therefore by the position, in our case. In a nonhomogeneous semiconductor, the average phonon density in state j is

$$(1) \quad \bar{n}_j = \frac{1}{4\pi^3 V} \int_{(V)} \int_{(V_{\vec{q}})} f_j(\vec{q}, \vec{r}, T) d\tau_{\vec{q}} d\tau_{\vec{r}},$$

where $f_j(\vec{q}, \vec{r}, T)$ is the Bose phonon distribution function, $d\tau_{\vec{q}}$ the volume element of $\vec{q}$ -space, $d\tau_{\vec{r}}$ the volume element of $\vec{r}$ -space and V the crystal volume. The phonon distribution function in the nonhomogeneous semiconductors is

$$(2) \quad f_j(\vec{q}, \vec{r}, T) = \left\{ \exp \frac{\hbar\omega_j(\vec{q}, \vec{r})}{k_B T} - 1 \right\}^{-1}.$$

The volume element of $\vec{q}$ -space can be written in the form

$$(3) \quad d\tau_{\vec{q}} = dS dq_{\perp},$$

where dS is an element of the constant energy surface in the $\vec{q}$ -space and $dq_{\perp}$ is normal to the two constant energy surfaces, $S_{\hbar\omega_j}$, and $S_{\hbar\omega_j + d(\hbar\omega_j)}$. The phonon energy depends on $\vec{q}$ and $\vec{r}$. Thus

$$(4) \quad d(\hbar\omega_j) = |\nabla_{\vec{q}}(\hbar\omega_j)| dq_{\perp} + |\nabla_{\vec{r}}(\hbar\omega_j)| d\vec{r}_{\perp}.$$

From (3) and (4) we obtain

$$(5) \quad d\tau_{\vec{q}} = \frac{d(\hbar\omega_j) - |\nabla_{\vec{r}}(\hbar\omega_j)| d\vec{r}_{\perp}}{|\nabla_{\vec{q}}(\hbar\omega_j)|} dS.$$

Introducing the relation (5) in (1) we have

$$(6) \quad \bar{n}_j = \frac{1}{4\pi^3 V} \int \int_{(V) (V_{\vec{q}})} f_j(\vec{q}, \vec{r}, T) \frac{d(\hbar\omega_j) - |\nabla_{\vec{r}}(\hbar\omega_j)| d\vec{r}_{\perp}}{|\nabla_{\vec{q}}(\hbar\omega_j)|} dS d\tau_{\vec{r}}.$$

We compare the expression (6) for $\bar{n}_j$ with the expression of the average phonon density as a function of the phonon density of states

$$(7) \quad n_j = \frac{1}{V} \int \int_{(\hbar\omega_j) = \text{cst } (V)} f_j(\vec{q}, \vec{r}, T) g(\omega_j, \vec{r}) d(\hbar\omega_j) d\tau_{\vec{r}}.$$

Thus, $g(\omega_j, \vec{r})$ is given by

$$(8) \quad g(\omega_j, \vec{r}) = \frac{1}{(2\pi)^3} \int_{(\hbar\omega_j) = \text{cst}} \frac{1 - |\nabla_{\vec{r}}(\hbar\omega_j)| |d\vec{r}_{\perp}/d(\hbar\omega_j)|}{|\nabla_{\vec{q}}(\hbar\omega_j)|} dS.$$

In a homogeneous semiconductor, the phonon energy $\hbar\omega_j$ is not dependent by position and $\nabla_{\vec{r}}(\hbar\omega_j) = 0$. Then, from relation (8) one obtains the known relation for the phonon density of states

$$(9) \quad g(\omega_j) = \frac{1}{(2\pi)^3} \int_{(\hbar\omega_j) = \text{cst}} \frac{dS}{|\nabla_{\vec{q}}(\hbar\omega_j)|}.$$

3. The Particular Case—Semiconductor having a Nonuniform Elastic Strain. We have applied the general theory for the phonon density of states in the semiconductors with position-dependent band structure to a semiconductor having a nonuniform elastic strain. First, we consider the case of the one-dimensional semiconductor. We note the homogeneous lattice by a and suppose that in nonhomogeneous crystal the distance from the atom in the position X_{i-1} to the atom in the position X_i is $a + bx_i$, where b is a constant.

Following the method of A.H. Marshak and K.M. van Vliet [5] we will consider a hypothetical homogeneous sample with the lattice constant $a+bx_i$, which is obtained from the nonhomogeneous semiconductor if the crystal structure between x_{i-1} and x_i is periodically repeated.

According, to the Gruneisen hypothesis we have the relation [16]

$$(10) \quad \frac{\Delta\omega}{\omega_0} = -\gamma \frac{\Delta V}{V_0},$$

where γ is the Gruneisen parameter. For a one-dimensional semiconductor having a nonuniform elastic strain we obtain

$$(11) \quad \omega(\vec{q}, x) = \omega_0(\vec{q}) \left(1 - \gamma \frac{bx}{a}\right).$$

Using the Debye approximation

$$(12) \quad \omega_0(\vec{q}) = C_s q,$$

where C_s is the longitudinal sound velocity, we have

$$(13) \quad \omega(\vec{q}, x) = C_s q \left(1 - \gamma \frac{bx}{a}\right).$$

From relations (8) and (13) it follows that

$$(14) \quad g(\omega, x) = \frac{q^2}{2\pi^2} \frac{1 + \left[\frac{a}{\gamma b q} \left(1 - \gamma \frac{bx}{a}\right) \frac{dq}{dx} - 1 \right]^{-1}}{\hbar C_s \left(1 - \gamma \frac{bx}{a}\right)}.$$

If the atoms number of semiconductor N is very high, then, the wave number q can be considered as a continuous function, like that the electron wave number k [13]

$$(15) \quad q = \beta \frac{2\pi}{a+bx},$$

where $\beta \in (-1, +1)$ for the first Brillouin zone.

From (14) and (15) we have

$$(16) \quad g(\omega, x) = \frac{\omega^2}{2\pi^2 C_s^2 \left(1 - \gamma \frac{bx}{a}\right)} \frac{1 - \left[\frac{2\pi a C_s \beta}{\gamma \omega x (a+bx)^2} \left(1 - \gamma \frac{bx}{a}\right) + 1 \right]^{-1}}{\hbar C_s \left(1 - \gamma \frac{bx}{a}\right)}.$$

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DENSITATEA DE STĂRI FONONICE ÎN SEMICONDUCTORI CU STRUCTURA DE BANDĂ DEPENDENTĂ DE POZIȚIE

(Rezumat)

Se studiază densitatea de stări fononice într-un semiconductor care are o structură de bandă neomogenă. Se stabilește o relație generală pentru densitatea de stări fononice care a fost apoi particularizată în cazul unui semiconductor supus unei tensiuni elastice neuniforme.

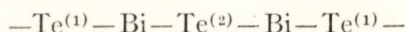
MODIFICATIONS INDUCED BY ANNEALING OF $\text{Bi}_2\text{Te}_{2.7}\text{Se}_{0.3}$ THIN FILMS

BY

ALEXANDRINA JEFLEA, SOFIA MELINTE and ȘT. VOLOSEK

The material structure based on Bi_2Te_3 and of its alloys may be described as a group of complexe layers, perpendicular to the trigonal axis (c axis in the hexagonal lattice) [1]. The same three types of quintets exist in the elementary cell.

The atoms of the separate layers are identical making a plane hexagonal lattice, the layers being successive



We can distinguish two types of chemical bonds—inside the layer and among layers. By comparing the distance between $\text{Bi}-\text{Te}^{(1)}$, $\text{Bi}-\text{Te}^{(2)}$, $\text{Te}^{(1)}-\text{Te}^{(1)}$ with the sum of the ionic radius and Van der Walls, one can attribute to the bond $\text{Bi}-\text{Te}^{(1)}$ and $\text{Bi}-\text{Te}^{(2)}$ the covalente nature while the bond $\text{Te}^{(1)}-\text{Te}^{(2)}$ (so between layers) is of Van der Walls type also imposed by Coulomb forces (the bond $\text{Bi}-\text{Te}^{(1)}$ has an ionic feature more evident than $\text{Bi}-\text{Te}^{(2)}$).

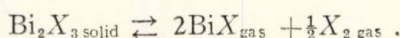
D r a b l e ' s and G o o d m a n ' s chemical bond theory is confirmed by the fact that gas absorbtion at the free surface of the crystal is reduced, so that chemical bonds being satisfied inside the material [2].

Bi_2Te_3 makes a continuous isomorphe solid alloy together with Bi_2Se_3 , these ones belonging to the same class of symmetry containing a lattice of very alike parameters and the atoms of Se and Te having similar dimensions and chemical properties. Te and Se covalent radius are 1.37 Å and 1.17 Å respectively. Theoretically, Se may replace any of Te atoms indifferent of $\text{Te}^{(1)}$ or $\text{Te}^{(2)}$ properties.

From a and c crystal parameter dependence as function of their structure, one can conclude the place occupied by Se. The a dependence of concentrations is uniform becoming lower while passing from Bi_2Te_3 to Bi_2Se_3 ; concerning c , the variation of this one shows a uniform lowering up to $\text{Bi}_2\text{Te}_2\text{Se}$ composition after which it grows exceeding Bi_2Te_3 value up to 67% of Bi_2Se_3 . This dependence suggests us that first Se will occupy the places $\text{Te}^{(1)}$ and then the places $\text{Te}^{(2)}$. The Van der Walls Se—Te-bond is weaker than Te—Te or Se—Se, so that c parameter grows until Se comes to occupy 50% of the places $\text{Te}^{(1)}$ while $\text{Se}^{(1)}-\text{Se}^{(1)}$ bond number versus

$\text{Se}^{(1)} - \text{Te}^{(1)}$ will be predominant, c beginning to lower. In such conditions, $\text{Bi}_2\text{Te}_{2.7}\text{Se}_{0.3}$ is supposed to have Se greater in $\text{Te}^{(2)}$ places.

The mass spectrometry [3] and mathematical calculations [4] have revealed that Bi_2Te_3 and Bi_2Se_3 evaporation is followed by a dissociation as in the relation



The total pressure temperature dependence of the saturated vapours is described by the relation

$$(1) \quad \lg P_{\text{total}} = -\frac{A}{T} + B,$$

where

$$(2) \quad A = 10\,443 \pm 444, \quad B = 11.054 \pm 0.228$$

and the components partials pressure are

$$(3) \quad p_{\text{Te}_2} = 0.25 p_{\text{BiTe}}, \quad p_{\text{BiTe}} = 0.8 p_{\text{total}}.$$

Taking into consideration the material composition and its properties concerning the evaporation, the problems resulted in obtaining thin films proved to be very difficult. After the dissociation process there will result four types of components BiTe, BiSe, Se_2 and Te_2 having very different vapour pressures. Among these components Se is evidenced possessing the property of sublimation, so a very low vapour pressure.

Except this one, Se adhesion coefficient at the clean substrate $\alpha = 0$. In these conditions, the simple evaporation method of the material is excluded. The method here used is based on the flash evaporation one.

The last one is characterized by the fact that the material arriving in the hot evaporator occurs in a discrete way, in a few quantity, the material charge being completely evaporated up to the next charge.

Let us analyse the first charge evaporation. The vapour pressure for Te and Se being inferior to that for BiTe_{gas} , Se and Te will evaporate before the other components do. Especially Se will evaporate much earlier than the others. The vapours of Se reaching the clean substrate will not adhere to this one. Next, BiTe, BiSe and Te_2 will evaporate condensing on the substrate resulting a layer without Se but with Bi and Te in excess, so nonstoichiometrical. When the next charge follows, the evaporation occurs in the same order. Now, the evaporated Se will find a layer where its adherence is good but the evaporation occurring at a high speed, the vapour fascicle of Se will adhere to the substrate realizing a layer saturation in Se, stopping the total vapour condensation. Thus, the evaporation-condensation mechanism will give a layer poor in Se. This difficulty can be eliminated either by controlling the arriving speed of the charges in the evaporator, their evaporation being partially superimposed, we mean the evaporation of the less volatile component of a charge could be made in parallel with Se evaporation of the next charge, or by the flash evaporation of the material in a Se atmosphere assured by Se evaporation from the source at a lower temperature [5].

The films thus obtained will have Se in excess. The spatial defects present in the bulky material are even more in thin films, the excess of Se will give clearance faults resulting donor states. This enhancement of the defect number in thin films makes to enhance the carrier concentration but at the same time lowers obviously the mobility.

The bulk material of $\text{Bi}_2\text{Te}_{2.7}\text{Se}_{0.3}$ delivered by ICPIAF Cluj-Napoca has the following features :

$$n = 1.2 \cdot 10^{19} \text{ cm}^{-3}, \quad \rho = 3.75 \cdot 10^{-3} (\Omega\text{cm})^{-1},$$

$$\alpha = 170 \mu\text{V/K}, \quad \mu = 1.302 \cdot 10^{-2} \text{ cm}^2/\text{Vs}.$$

The thin films obtained by flash evaporation in a Se atmosphere with thickness measured in have generally a lower resistivity in comparison with that one of the bulk material $1.2 \dots 1.9 \cdot 10^{-3} \Omega\text{cm}$. These films also present an unsatisfying Seebeck factor ($40 \mu\text{V/K}$) close to that theoretically established in case of degeneration. Characteristical to the degeneration condition is also the carrier concentration in these samples. These ones were submitted to a thermal treatment in the air at growing temperatures. The variation of the thin films resistance after thermal treatment up to 300°C is not significant but at temperatures between 300°C and 400°C the film resistance is considerably increased (Fig. 1). This modification shows the lowering of Se atom number interstitial defects by their sublimation.

On the walls of the thermal treatment house a Se layer is deposited which after several treatments becomes visible with the naked eye. We also admit that in case of rebaking, a structure symmetry growing will occur in the bulk material and in the thin layer evidenced by X-ray diffraction. Except the resistance increment, the other thermoelectrical parameters are modified. Thus, for a sample untreated by thermal method, α coefficient has the temperature dependence as in Fig. 2 curves *a* and after a treatment

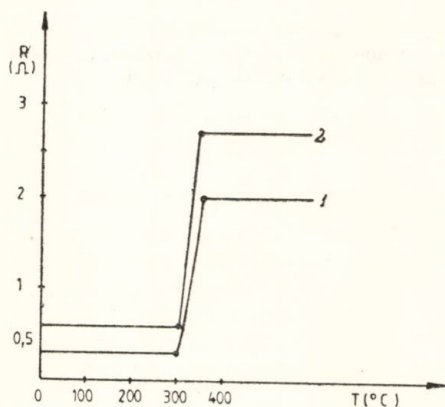


Fig. 1 — Resistance of two thin films v.s. temperature annealing.

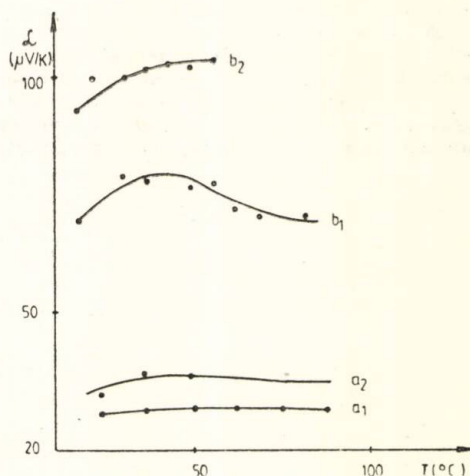


Fig. 2 — Temperature dependence of Seebeck factor for two films: *a*—before annealing; *b*—after annealing.

at 350°C has the dependance given by the curves *b*. For the same sample, a modification of the carrier concentration between $3.887 \cdot 10^{21} \text{ cm}^{-3}$ during the treatment and $1.007 \cdot 10^{20} \text{ cm}^{-3}$ after the treatment has been found. It is obvious that the carrier concentration is still greater than the propitious one of the bulk material. This high concentration is due to the structure defects more numerous than in the bulk material. Their diminishing is obtained by growing film thickness and film crystallisation degree or by repeating the thermal treatments at slow cooling.

Conclusions after thin film analyses will be checked up in obtaining thicker films in the above experimental conditions.

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MODIFICĂRI INDUSE ÎN URMA TRATAMENTELOR TERMICE ÎN STRATURILE SUBȚIRI DE $\text{Bi}_2\text{Te}_{2.7}\text{Se}_{0.3}$

(Rezumat)

Problemele obținerii straturilor subțiri termoelectrice din combinațiile pe bază de Bi_2Te_3 sînt deosebite, deoarece acești compuși disociază la evaporare în elementele componente. Pe lângă aceste fenomene, apare și problema evaporării Se, stratul obținut prin evaporare flash fiind sărăcit în Se față de materialul masiv de bază. Suplimentarea Se în timpul depunerii stratului duce la obținerea unui strat nestoichiometric.

În această Notă se cercetează comportarea termoelectrică a straturilor obținute din $\text{Bi}_2\text{Te}_{2.7}\text{Se}_{0.3}$ în funcție de durata și temperatura tratamentelor termice. Parametrii studiați au fost coeficientul Seebeck, mobilitatea purtătorilor și concentrația de purtători.

EFFETS DE LA SUBSTITUTION PARTIELLE DES IONS DE LANTHANE PAR DES IONS DE BISMUTH SUR LES PERFORMANCES DES PIÉZOCÉRAMIQUES DE TYPE PZT APPARTENANT AU SYSTÈME FeLaBi

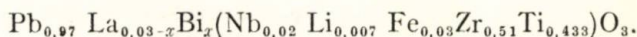
PAR

IRINA ANTONESCU, POMPIIU NICOLAU et C. TÂNĂSOIU

1. Introduction. On a réalisé quatre matériaux piézocéramiques nouveaux, à base de zirco-titanate de plomb modifié, donc de type PZT, appartenant au système FeLa [1]. On a tâché d'obtenir une activité piézoélectrique intense aussi qu'une dépolarisation réduite, de sorte que ceux-ci puissent être utilisés comme résonateurs de bande large et de puissance moyenne [1].

Pour réaliser aussi l'abaissement de la température de frittage tout comme l'amélioration du contrôle de la volatilisation des ions de plomb, on a remplacé une partie des ions de lanthane par des ions de bismuth dans le composé du système FeLa ayant la proportion maximale de Fe^{3+} et de La^{3+} (le composé FeLa3). On a obtenu ainsi trois nouveaux matériaux piézocéramiques, appartenant à un autre système, nommé FeLaBi, caractérisés par des propriétés diélectriques et piézoélectriques compétitives à celles des matériaux similaires, appartenant à la classe des piézocéramiques à facteur de qualité mécanique réduit.

2. Matériaux et mesures expérimentales. Les composés du système FeLaBi sont stoechiométriques, résultés par la substitution d'une partie des ions de La^{3+} , des sites dodécaédriques du composé FeLa3, par des ions de Bi^{3+} . Donc leur formule est la suivante :



En mettant $x=0$; $x=0,01$; $x=0,02$; $x=0,03$ on obtient les composés notés FeLaBi0, FeLaBi1, FeLaBi2 et, respectivement, FeLaBi3.

On a préparé ces matériaux par une technique céramique appropriée du mélange d'oxydes des métaux respectifs. Il faut remarquer le frittage en l'air, à de basses températures (environ 1000°C). La technologie et les méthodes de mesures ont été présentées antérieurement [2].

Les mesures expérimentales ont compris : analyse structurale par diffraction des rayons X, détermination de la porosité des matériaux, observations métallographiques, détermination de la température Curie, des propriétés diélectriques à basse fréquence (ϵ_{33}^T et $\text{tg} \delta$ à 1 592 Hz) et des

propriétés piézoélectriques (facteur de couplage piézoélectrique k_p , constante de la fréquence N_p , constantes piézoélectriques d_{31} et g_{31} , compliance élastique s_{11}^E etc.).

3. Résultats et discussions. Les analyses structurales par diffraction des rayons X ont confirmé la formation des matériaux monophasiques, dont la tétragonalité varie à mesure que la proportion des ions de Bi^{3+} sur les sites dodécaédriques augmente (par exemple les valeurs du rapport c/a sont 1,0199 pour FeLaBi0 et 1,0227 pour FeLaBi2).

Les observations métallographiques ont mis en évidence l'accroissement des dimensions des cristallites à partir des valeurs inférieures à $5\text{ }\mu\text{m}$ — au cas du matériau FeLaBi0 — jusqu'à plus de $5\text{ }\mu\text{m}$ et même $10\text{ }\mu\text{m}$ au cas du matériau FeLaBi1 . L'augmentation de la proportion des ions de Bi^{3+} au-dessus de 1 atome % ne conduit pas à des variations notables de la granulation, les cristallites étant bien empaquetées, d'environ $5\text{ }\mu\text{m}$.

On a constaté que la porosité des échantillons frittés à des températures différentes présente un minimum à une certaine température, T_0 , variable selon la composition. Pour distinguer les frittages qui ont conduit à des porosités égales, mais ont été faits à des températures différentes, dans la représentation graphique des propriétés en fonction de la porosité on a utilisé des figures géométriques blanches (Δ $\square$ ∇) et, respectivement, noires. Donc les valeurs obtenues par frittage à des températures situées au-dessous de T_0 ont été représentées par figures géométriques blanches, tandis que celles obtenues par frittage à des températures situées au-dessus de T_0 , par figures géométriques noires. De sorte on a mis en évidence l'influence de la granulation sur les propriétés diélectriques et piézoélectriques.

Les variations des grandeurs diélectriques, constante diélectrique $\epsilon_{33}^T/\epsilon_0$ et tangente de l'angle de pertes $\text{tg } \delta$, aussi que celles des grandeurs piézoélectriques, facteur de couplage k_p , constante de la fréquence dans le mode radial de vibration N_p , et constante de charge piézoélectrique d_{31} , ont été représentées dans les Figs. 1—5. On y voit que :

a) Au cas de températures de frittage inférieures à T_0 , la constante diélectrique de tous les matériaux s'accroît à mesure que la porosité baisse. Dans le domaine des températures de frittage supérieures à T_0 , la courbe de la variation de la constante diélectrique en fonction de la porosité présente un maximum correspondant à une dimension optimale des cristallites. Elle est donc déterminée par la granulation du matériau (Fig. 1).

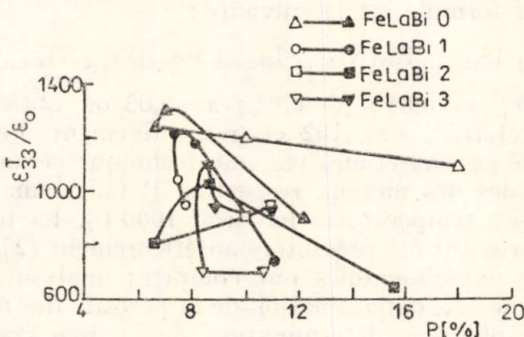


Fig. 1

Les valeurs de la constante diélectrique des composés du système sont situées entre 1 300 (FeLaBi0) et 900 (FeLaBi3). Il en résulte que celle-ci diminue à mesure que la proportion des ions de Bi^{3+} augmente, l'effet de la composition prévalant l'effet de la dimension des cristallites.

b) Les pertes diélectriques de tous les composés diminuent à mesure que la porosité s'accroît (Fig. 2). On y voit qu'à une même valeur de la porosité correspondent deux valeurs différentes de la tangente de l'angle de pertes, l'hystérésis de cette grandeur en fonction de la porosité en étant évident (Fig. 2).

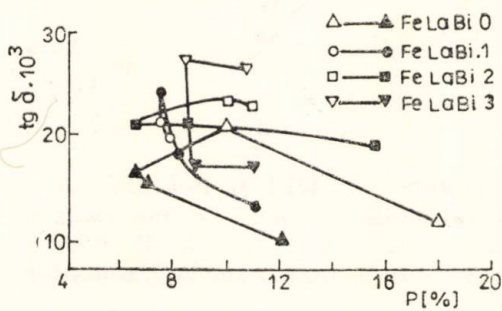


Fig. 2

La réduction des pertes diélectriques à basse fréquence à mesure que la porosité augmente est due à la conductibilité électrique, moindre au cas des porosités plus grandes. Les pertes diélectriques diminuées, obtenues par frittages à des températures supérieures à T_0 , peuvent être expliquées par l'amoindrissement de la polarisation interfaciale au cas des granulations caractérisées par de larges cristallites.

Les composés ayant 1–2 ions de Bi^{3+} % sur les sites dodécaédriques ont la tangente de l'angle de pertes plus grande ($>20 \cdot 10^{-3}$) par rapport à ceux qui ont sur les mêmes sites uniquement du La^{3+} (FeLaBi0) ou du Bi^{3+} (FeLaBi3). Les derniers ont $\text{tg } \delta$ de l'ordre $16 \cdot 10^{-3}$.

c) La constante de charge piézoélectrique d_{31} augmente de façon monotone en fonction de la porosité jusqu'à la valeur minimale de cette dernière.

Dès lors la constante d_{31} présente un maximum bien exprimé, à une valeur de la porosité qui correspond au maximum de la constante diélectrique (Fig. 3).

Les substitutions jusqu'à 1 ion de Bi^{3+} % sur les sites du La^{3+} conduisent à des valeurs maximales de la constante de charge piézoélectrique ($d_{31} = 166 \cdot 10^{-12}$ m/V), celles-ci étant toutefois inférieures aux valeurs obtenues au cas des composés du système FeLa [1].

d) La constante de la fréquence dans le mode radial de vibration N_p présente hystérésis en fonction de la porosité uniquement au cas des composés ayant tant du La^{3+} que du Bi^{3+} sur les sites dodécaédriques. La vitesse de variation de cette constante en fonction de la porosité est minimale au cas des composés FeLaBi0 (sans Bi^{3+}) et FeLaBi2 (qui a 2 ions de Bi^{3+} % sur les sites dodécaédriques) (Fig. 4). Les vitesses de variation plus grandes

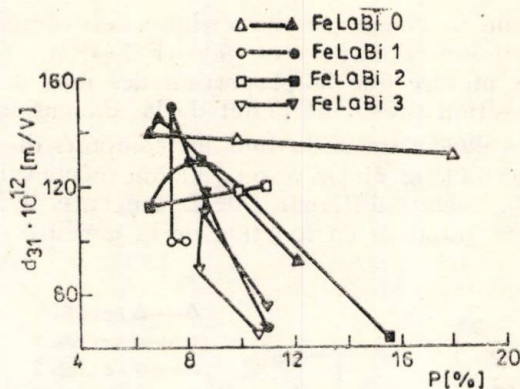


Fig. 3

des deux autres composés, FeLaBi1 et FeLaBi3, peuvent être attribuées au fait que la densification est presque constante dans un domaine suffisamment étendu de températures de frittage, les différences entre les porosités qui en résultent étant réduites, tandis que la granulation varie considérablement.

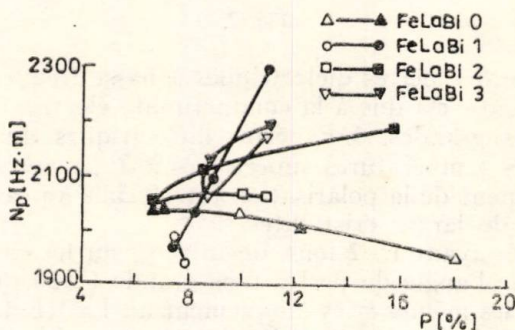


Fig. 4

La constante de la fréquence s'accroît de 1950 Hz.m à 2150 Hz.m à mesure que la proportion des ions de Bi^{3+} sur les sites dodécaédriques augmente.

e) Le facteur de couplage piézoélectrique, k_p , présente un comportement hystérétique en fonction de la porosité (Fig. 5). Tant au cas des frittages effectués aux températures inférieures à T_0 , ainsi qu'au cas de ceux effectués à des températures supérieures à T_0 , le facteur de couplage s'accroît à mesure que la porosité diminue. Dans le domaine des températures de frittage supérieures à T_0 , tous les matériaux ayant tant du Bi^{2+} que du La^{3+} sur les sites dodécaédriques présentent des maxima du facteur de couplage k_p en fonction de la porosité pour des valeurs de celle-ci situées entre 7% et 9%. Il en résulte que la substitution du lanthane par du bismuth sur les sites dodécaédriques est avantageuse tant que les deux types d'ions coexistent. Les matériaux dont la substitution du plomb des sites dodécaédriques a été faite unique-

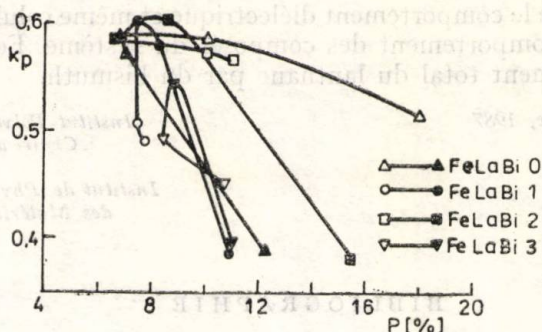


Fig. 5

ment par du La^{3+} (FeLaBi0) ou par du Bi^{3+} (FeLaBi3) ont des activités piézoélectriques diminuées par rapport aux autres composés du système ($0,54 \leq k_p \leq 0,59$).

Les températures Curie des composés du système FeLaBi sont supérieures à 240°C , ce que leur assure une bonne stabilité thermique. Les valeurs de la température Curie augmentent à mesure que la proportion des ions de Bi^{3+} s'accroît, mais elles restent inférieures à celles obtenues au cas du système FeLa [1].

4. Conclusions. 1. On a réalisé trois nouveaux matériaux piézocéramiques à base de zirco-titanate de plomb, à substitutions multiples tant sur les sites octaédriques, que sur celles dodécaédriques, en utilisant une technologie céramique simplifiée, dont le frittage a été fait en l'air à des températures relativement basses ($\sim 1\,000^\circ\text{C}$).

2. On a étudié l'influence de la température de frittage sur les propriétés diélectriques et piézoélectriques. On a trouvé que les frittages effectués à des températures situées dans un domaine d'environ 50°C au-dessus de la température correspondante à la porosité minimale T_0 , différente d'un matériau à l'autre, conduisent à l'amélioration des valeurs de la constante diélectrique et, par conséquent, de la constante de charge piézoélectrique d_{31} , sans pourtant diminuer l'activité piézoélectrique, mesurée par le facteur de couplage k_p .

3. On a essayé de mettre en évidence l'influence de la substitution partielle des ions de lanthane, des sites dodécaédriques, par des ions de bismuth, substitution avantageuse pour la technologie parce qu'elle réduit la température de frittage.

On a trouvé que cette substitution diminue de façon insignifiante l'activité piézoélectrique tant que les deux types d'ions, La^{3+} et Bi^{3+} , coexistent sur les sites dodécaédriques, mais elle la réduit de façon importante quand le lanthane est remplacé totalement par le bismuth. De plus, en ce cas-ci les constantes diélectrique et de charge piézoélectrique diminuent considérablement par rapport aux valeurs obtenues au cas du système FeLa [1]. En même temps les pertes diélectriques augmentent, tout aussi que la constante N_p de la fréquence. Ça veut dire que la compliance élastique s_{11}^E diminue elle aussi. Il en résulte que la substitution partielle du lanthane par du bismuth, sur les sites dodécaédriques, avantageuse pour la techno-

logie, endommagement le comportement diélectrique et même celui piézoélectrique par rapport au comportement des composés du système FeLa, surtout au cas du remplacement total du lanthane par du bismuth.

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EFECTELE SUBSTITUȚIEI PARȚIALE A IONILOR DE LANTAN PRIN BISMUT ASUPRA PERFORMANȚELOR PIEZOCERAMICILOR DE TIP PZT APARTININD SISTEMULUI FeLaBi

(Rezumat)

Într-unul din materialele sistemului FeLa, sistem bazat pe zirco-titanatul de plumb modificat, și anume în cel cu procentul maxim de Fe^{3+} și La^{3+} , s-au substituit parțial ionii de La^{3+} prin ioni de Bi^{3+} , în vederea coboririi temperaturii de sinterizare.

S-a urmărit influența temperaturii de sinterizare asupra proprietăților dielectrice și piezoelectrice ale compuşilor noi realizați. De asemenea au fost comparate performanțele acestor compuşii cu cele ale materialului de bază.

LES PROPRIÉTÉS DES PIÉZOCÉRAMIQUES DE TYPE PZT À SUBSTITUTIONS MULTIPLES DE Fe^{3+} ET DE La^{3+}

PAR

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1. Introduction. Les piézocéramiques à base de zirco-titanate de plomb (de type PZT) vont remplacer le titanate de baryum (BT), dans la mesure que leur technologie assure le contrôle de la volatilisation du plomb pendant le frittage, par des dépenses minimales.

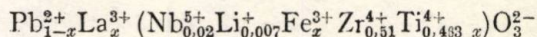
Les performances des piézocéramiques de type PZT sont supérieures à celles de type BT en étant influencées d'une façon importante par le type et la proportion des substitutions utilisées. Quand le but est l'accroissement de l'activité piézoélectrique on peut faire soit des substitutions dont la valence est plus grande que celle des ions de Ti^{4+} , sur les sites octaédriques, soit des substitutions dont la valence est différente par rapport à celle du Pb^{2+} , sur les sites dodécaédriques. Il en résulte des matériaux dont l'activité piézoélectrique est intense et le facteur de qualité mécanique est réduit. Leur vieillissement est réduit aussi, mais les pertes diélectriques sont relativement grandes, tandis que les valeurs du champ coercitif sont petites. En vue de l'accroissement du champ coercitif et, par conséquent, de la difficulté de la dépolarisation électrique, on fait des substitutions dont la valence est inférieure à celle des ions de Ti^{4+} , sur les sites octaédriques [1].

Ces matériaux, dont l'activité piézoélectrique est intense et les champs coercitifs sont améliorés, peuvent être utilisés comme des résonateurs de bande large et de puissance moyenne.

Dans cet ouvrage on soumet à l'étude quatre matériaux piézocéramiques nouveaux, obtenus par une technologie simplifiée, dont l'activité piézoélectrique est intense et les dépolarisations thermique et électrique sont améliorées.

2. Préparation et méthodes de mesurage. Les quatre nouveaux composés ont été obtenus en modifiant le matériau NbLi_2 , décrit antérieurement [2], dont les performances ont été trouvées très bonnes par rapport à la classe des matériaux piézocéramiques à activité intense. Ce matériau-ci est le résultat du remplacement de 2,7% ions Ti^{4+} par 2% ions de Nb^{5+} et 0,7% ions de Li^+ dans la solution solide de base $\text{Pb}(\text{Zr}_{0,51}\text{Ti}_{0,49})\text{O}_3$. En vue de l'amointrissement de la dépolarisation électrique, dans ce matériau-ci une autre partie des ions de Ti^{4+} a été remplacée par des ions de Fe^{3+} . Les quatre nouveaux composés correspondent à la substitution du Ti^{4+} par Fe^{3+} dans les proportions suivantes: 0%; 1%; 2%; 3%. Pour obtenir des composés

stoechiométriques, une certaine partie d'ions de Pb^{2+} a été remplacée par des ions de La^{3+} , la formule générique étant la suivante:



Les composés notés FeLa0, FeLa1, FeLa2 et FeLa3 correspondent aux substitutions dont les valeurs sont : $x=0$; $x=0,01$; $x=0,02$ et $x=0,03$.

La préparation des matériaux a été faite par la technique céramique ordinaire, du mélange d'oxydes des métaux respectifs, de pureté, en général, p.a. Les principales étapes de cette technologie, la technique de la polarisation électrique et les méthodes de mesurage ont été exposées antérieurement [2]. Notons toutefois une particularité importante de cette technologie : le frittage en l'air, à des températures relativement basses (environ 1100°C), importante en vue de l'industrialisation.

Les mesures expérimentales ont compris :

- 1) analyse structurelle par diffraction des rayons X ;
- 2) détermination de la densité des échantillons ;
- 3) observations métallographiques pour caractérisation morphologique (dimensions des cristallites et distribution des pores) ;
- 4) détermination de la température Curie ;
- 5) étude du comportement diélectrique à basse fréquence (la constante diélectrique ϵ_{33}^T et $\text{tg } \delta$ à 1592 Hz) ;

6) étude des propriétés piézoélectriques (le facteur de couplage planaire k_p , la constante de la fréquence N_p^E pour le mode radial de vibration, les constantes piézoélectriques d_{31} et g_{31} , la compliance élastique s_{11}^E etc.), par la méthode dynamique, de résonance [2].

3. Résultats expérimentaux. L'analyse structurelle par diffraction des rayons X a confirmé la formation des composés monophasiques, dont la tétragonalité diminue à mesure que la proportion x de la substitution double de Fe^{3+} et de La^{3+} augmente. On a obtenu des valeurs $c/a=1,0232$ pour $x=0$ et $c/a=1,0199$ pour $x=0,03$. De plus, la constante diélectrique de tous les composés du système FeLa augmente après le traitement de polarisation électrique. Ce fait confirme lui aussi la tétragonalité des structures analysées [3].

La variation de la densité des échantillons en fonction de la température de frittage présente un maximum à une certaine température T_0 , différente d'un matériau à l'autre. À la densité maximale correspond la porosité minimale. L'augmentation de la température de frittage au-dessus de la valeur T_0 conduit en même temps à des modifications de la densité et de la granulation. Dans les Figs. 1 jusqu'à 5 sont présentées les variations des principaux paramètres diélectriques et piézoélectriques en fonction de la porosité, cette-ci étant une mesure de la température de frittage. Pour mettre en évidence l'influence de l'accroissement des dimensions des cristallites sur les paramètres diélectriques et piézoélectriques, dans la représentation graphique on a distingué les traitements qui ont conduit à des porosités égales, mais ont été effectués à des températures inférieures et, respectivement, supérieures à T_0 : les frittages effectués à des températures moindres que T_0 , ont été représentés par des figures géométriques blanches, tandis que ceux effectués à des températures plus grandes que T_0 , par des figures géométriques noires.

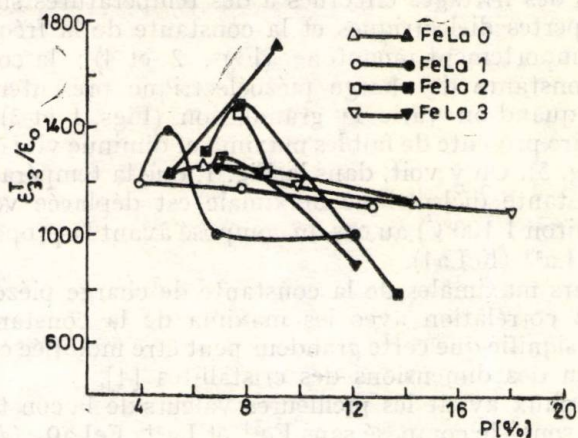


Fig. 1

On a trouvé, par observations métallographiques, que les dimensions des cristallites diminuent à mesure que la proportion de la substitution double de Fe^{3+} et de La^{3+} augmente.

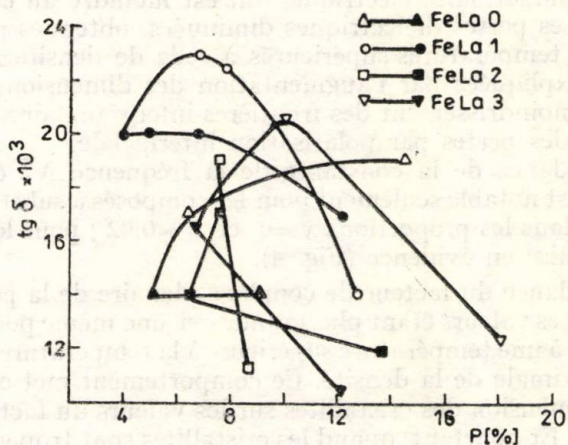


Fig. 2

Les températures Curie des composés du système FeLa sont comprises entre 330° et 396°C , ce que leur assure une meilleure stabilité thermique, dans un domaine élargi (environ 150°C) au-dessus de la température ambiante.

En ce que concerne le comportement diélectrique et piézoélectrique, on a constaté que :

a) au cas des frittages effectués à des températures inférieures à T_0 , tous les paramètres caractéristiques présentent des variations monotones, c'est-à-dire ils augmentent à mesure que la porosité diminue ;

b) au cas des frittages effectués à des températures supérieures à T_0 , seulement les pertes diélectriques et la constante de la fréquence N_P^E conservent leur comportement monotone (Figs. 2 et 4); la constante diélectrique et la constante de charge piézoélectrique présentent des maxima bien exprimés, quand on varie la granulation (Figs. 1 et 3). Le facteur de couplage planaire présente de faibles maxima et diminue vite quand la porosité augmente (Fig. 5). On y voit, dans la Fig. 1, que la température qui correspond à la constante diélectrique maximale est déplacée vers des valeurs inférieures (environ 100°C) au cas du composé ayant la proportion minimale de Fe^{3+} et de La^{3+} (FeLa1).

Les valeurs maximales de la constante de charge piézoélectrique, d_{31} , se trouvent en corrélation avec les maxima de la constante diélectrique (Fig. 1), ce que signifie que cette grandeur peut être modifiée convenablement par la variation des dimensions des cristallites [4].

Les matériaux ayant les meilleures valeurs de la constante de charge piézoélectrique sont : le composé sans Fe^{3+} et La^{3+} , FeLa0 , ($d_{31} = -168.10^{-12}$ m/V) et celui qui a 2% ions de Fe^{3+} et de La^{3+} , FeLa2 ($d_{31} = -180.10^{-12}$ m/V).

Au cas de tous les composés, les pertes diélectriques diminuent à mesure que la porosité augmente, en étant évident l'hystérésis de cette grandeur en fonction de la porosité (Fig. 2), à une même valeur de la porosité correspondent deux valeurs différentes des pertes diélectriques. La réduction des pertes diélectriques à basse fréquence à mesure que la porosité augmente est due à la conductibilité électrique, qui est moindre au cas des porosités plus grandes. Les pertes diélectriques diminuées, obtenues par des frittages effectués à des températures supérieures à celle de densification maximale, peuvent être expliquées par l'augmentation des dimensions des cristallites et donc par l'amointrissement des frontières intergranulaires, ce que conduit à la réduction des pertes par polarisation interfaciale.

La dépendance de la constante de la fréquence N_P^E de la dimension des cristallites est notable seulement pour les composés à substitutions doubles (Fe^{3+} et La^{3+}) dans les proportions $x=0$ et $x=0,02$; pour les autres elle ne peut pas être mise en évidence (Fig. 4).

La dépendance du facteur de couplage planaire de la porosité présente de l'hystérésis, ses valeurs étant plus grandes si une même porosité a été obtenue par frittage à une température supérieure à la température correspondante à la valeur maximale de la densité. Ce comportement met en évidence l'influence de la dimension des cristallites sur les valeurs du facteur de couplage piézoélectrique. Et pourtant, quand les cristallites sont trop grandes (environ 15...20 μm), le facteur de couplage piézoélectrique diminue, au cas de tous les composés. Donc, d'une part on remarque l'accroissement du facteur de couplage piézoélectrique à mesure que les dimensions des cristallites augmentent, d'autre part les matériaux à cristallites trop grandes ont, pour ce facteur, des valeurs inférieures. Ce comportement peut être expliqué par l'effet compétitif de la dimension des cristallites et des microfissures dues aux tensions mécaniques intercristallines, plus grandes à mesure que les cristallites augmentent [5]. Au fur et à mesure que les cristallites s'accroissent, leur séparation est plus grande et la charge spatiale créée sur les frontières de granulation augmente, ce que conduit à l'abaissement du facteur de couplage piézoélectrique [6].

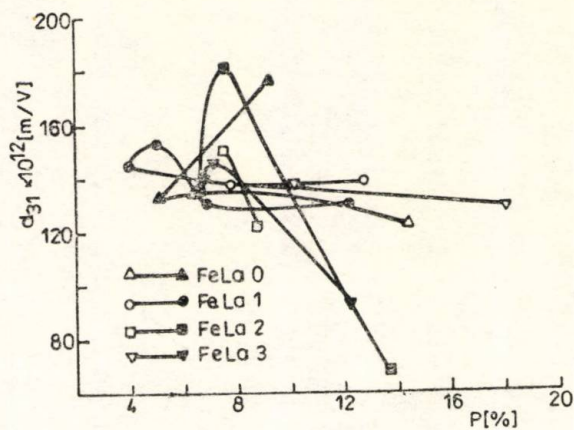


Fig. 3

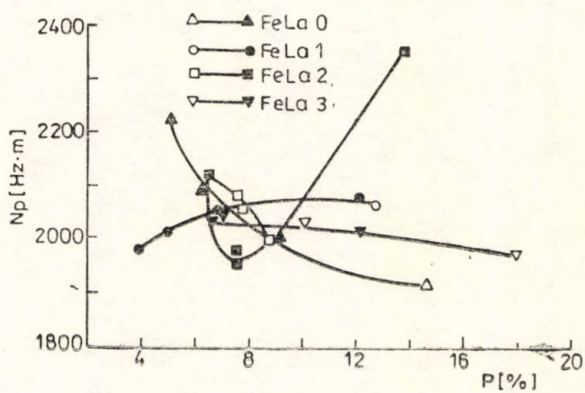


Fig. 4

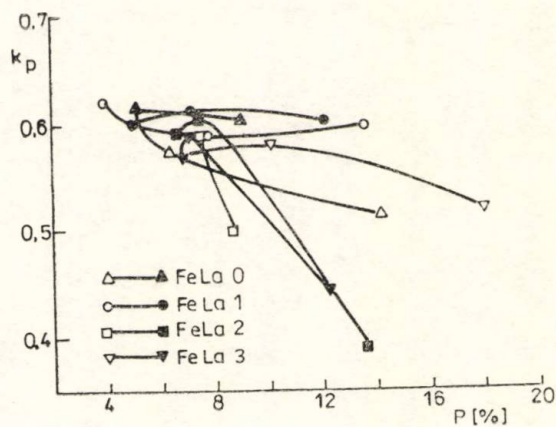


Fig. 5

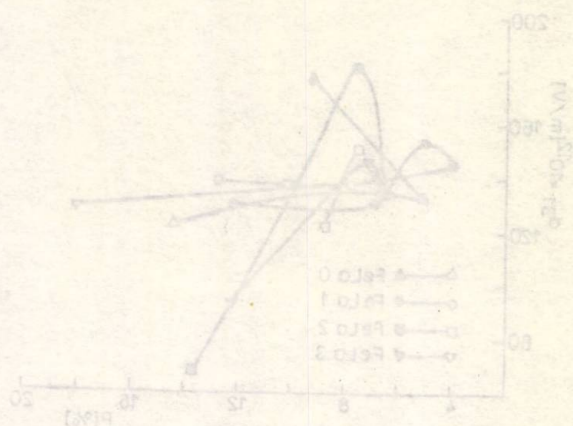


Fig. 1

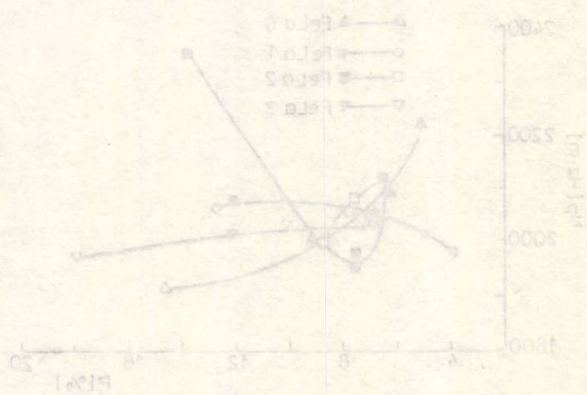


Fig. 2

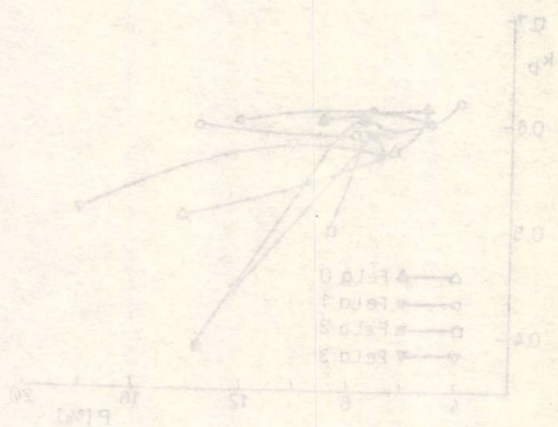


Fig. 3

Dans la Fig. 5 on remarque, pour les frittages effectués à des $T > T_0$, de faibles variations du facteur de couplage planaire en fonction de la porosité, surtout au cas des matériaux sans Fe^{+3} et La^{+3} (FeLa0) et avec la moindre proportion de Fe^{+3} et de La^{+3} (FeLa1). Ce fait peut être utilisé comme facilité technologique : le domaine élargi des températures de frittage permet, au cas de ces deux matériaux, des traitements finaux à des températures supérieures à la température de densification maximale, ce que conduit à des dimensions des cristallites convenables à l'obtention des valeurs maximales de la constante diélectrique et, par conséquent, de la constante de charge piézoélectrique d_{31} .

4. Conclusions. 1. On a réalisé quatre nouveaux matériaux piézocéramiques, à base de zirco-titanate de plomb, à substitutions multiples, en utilisant une méthode originale, basée sur la technologie céramique, dont le frittage a été fait en l'air, à des températures relativement basses (environ $1\ 100^\circ\text{C}$), adaptable à l'industrialisation.

2. On a mis en évidence l'influence de la morphologie des cristallites sur les propriétés du matériau :

a) on a constaté que la constante diélectrique, tout comme la constante de charge piézoélectrique et le facteur de couplage planaire présentent des valeurs maximales pour certaines dimensions des cristallites ;

b) on a trouvé que la valeur du facteur de couplage piézoélectrique est le résultat de l'effet compétitif de l'accroissement des dimensions des cristallites, qui le renforce, et de l'engendrement des microfissures dues aux tensions mécaniques intercristallines, qui l'affaiblit. Les dernières sont d'autant plus importantes que les cristallites sont plus grandes ;

c) on a montré que les pertes diélectriques présentent hystérésis en fonction de la porosité, ce comportement mettant en évidence le fait que le mécanisme prédominant de ces pertes-ci est la mobilité des parois des domaines, dont le nombre diminue à mesure que la dimension des cristallites augmente.

On a déterminé les conditions dans lesquelles on peut obtenir les granulations appropriées à l'amélioration de la constante diélectrique, aussi que de la constante de charge piézoélectrique, sans pourtant diminuer le facteur de couplage planaire (les composés FeLa1 et FeLa0).

3. On a constaté que la substitution double de Fe^{+3} et de La^{+3} est avantageuse tant du point de vue piézoélectrique que diélectrique : les matériaux du système FeLa présentent, à la température ambiante, des activités piézoélectriques intenses, étant caractérisés par des facteurs de couplage planaire supérieurs à 0,60 et par des constantes de charge piézoélectrique, d_{31} , meilleurs que $-160 \cdot 10^{-12}$ m/V. Les constantes diélectriques relatives dépassent 1 300, sans que les valeurs des pertes diélectriques deviennent excessives ($\text{tg } \delta < 17 \cdot 10^{-3}$). Les composés du système FeLa se sont révélées compétitifs à n'importe quel matériau de la classe des piézocéramiques à haut facteur de couplage piézoélectrique.

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PROPRIETĂȚILE PIEZOCERAMICILOR DE TIP PZT CU SUBSTITUȚII MULTIPLE DE Fe ȘI La

(Rezumat)

S-au realizat patru materiale piezoceramice noi, pe bază de zirco-titanat de plumb (de tip PZT), cu substituții multiple, folosind o tehnologie ceramică simplificată. S-a studiat influența substituției duble de Fe-La și a tratamentului termic asupra proprietăților dielectrice și piezoelectrice ale noilor compusi. Performanțele lor s-au dovedit superioare celor ale materialului de bază și comparabile cu ale celor mai buni compusi din categoria piezoceramicilor cu factor de cuplaj electromecanic înalt și factor de calitate mecanic redus.

RECENZII

BOOK REVIEWS

РЕЦЕНЗИИ

MICHAEL J. GRIMBLE and MICHAEL A. JOHNSON, **Optimal Control and Stochastic Estimation: Theory and Applications** (two volumes), John Wiley & Sons, 1988, Chichester-New York-Brisbane-Toronto-Singapore, XX + XVIII + 997 pp.

The two volumes under review present a self-contained and comprehensive overview of linear optimal control and stochastic estimation theory. Contents: 1. *A Time-Domain Analysis of Deterministic Optimal Control Problems*; 2. *Frequency-Domain of Deterministic Optimal Control Problems: A Wiener-Hopf Approach*; 3. *The Asymptotic Behaviour of Root Loci*; 4. *The Design of Optimal Control Systems by Cost Weight Selection*; 5. *The Maximal Accuracy of Linear Optimal Regulators and Related Topics*; 6. *Optimal Control Structures*; 7. *A Deterministic Industrial Control System Study*; 8. *Time Domain Analysis of Filtering and Smoothing Problems*; 9. *Frequency Domain Analysis of Filtering and Smoothing Problems*; 10. *Time-Domain Analysis of the Optimal Stochastic Linear Control Problem*; 11. *The Frequency Domain Analysis of the Stochastic Control Problem*; 12. *Optimal Self-Tuning Control Systems*; 13. *Stochastic Industrial Control Systems*.

Each chapter contains a reference list and each volume an index.

The first volume (Ch. 1-7) is mainly devoted to the *deterministic case*, while the second one (Ch. 8-13) is concerned with a *systematic study of the stochastic aspects* of the subject.

The authors — two very experienced researchers in the area — succeeded in making a very clear and convincing presentation of many new concepts, results and methods in linear control and stochastic estimation. It should be noted that many of the results included have never been collected into a single monograph and some approaches are completely new. See for instance Ch. 9. Ch. 7 and 13 are of particular interest since both of them clearly illustrate the power of the abstract theory in solving problems of great practical importance: the design of a shape-control system for a cold-rolling mill — Ch. 7 — and the ship-positioning design procedure — Ch. 13.

The book is very useful to post-graduate industrial engineers, graduate students in both mathematics and engineering, as well as, to undergraduate students.

I. I. Vrabie

B. A. KUPERSHMIT, **Elements of Superintegrable Systems, Basic Techniques and Results**, Mathematics and its Applications; D. Reidel Publishing Company, Dordrecht-Boston-Lancaster-Tokyo, 1987, XVI + 187 pp.

Integrable system theory is an example of a field where a most unusual number of disciplines come together. It involves: algebraic geometry via abelian varieties and flexes; Lie algebras via both symmetry considerations and orbits of the co-adjoint representation; infinite dimensional Lie groups in the form of loop groups; functions of a complex variable via the Riemann-Hilbert boundary value problem; partial differential equations via the δ -Neumann problem and also pseudo-differential operators; the representations theory of the symmetric and general linear group; calculus of variations via the formal variational calculus approach initiated by Gelfand; quantum mechanics, fluid mechanics, dynamics of rigid bodies, statistical physics, (a number of important model equations in these fields of physics are integrable).

The even parts of mathematics constitute (roughly speaking) ordinary algebra, analysis, geometry,... This parts concern just „non-super“ integrable systems. However, these substantial parts have super-counterparts where both ordinary commutative functions occur and odd functions which anticommute. Thus we have super-Lie-algebras, superspaces, supermanifolds, supergroups, superintegrable systems. The odd parts correspond to fermions in physics, and the even correspond to bosons.

In the first part, *Classical Superintegrable Systems*, is given the language of the variational calculus with anticommutative variables and the basic grammatical constructions of this language. The main objects in this section are: differential forms, variational and Fréchet derivatives, adjoint operators, superHamiltonian structures, stable Lie superalgebras over commutative superalgebras, generalized two-cocycles on stable Lie superalgebras, supertrace, the differential of a matrix pseudo-differential operator, the pseudo-differential Lax operator, the Lax equations. The main results are: full description of both the Kernel and the Image of the variational operator, an algebraic criterion for a map to be canonical, a one-to-one correspondence between linear Hamiltonian structures and stable Lie superalgebras, formule for the supertrace of the Residue of the differential of a power of a matrix pseudo-differential operator, a construction of an infinite common set of conservation laws for Lax derivations.

In the second section, *Lie Algebras, Kortweg-de Vries Equations and Bi-SuperHamiltonian Systems*, three different generalizations of the Kortweg-de Vries equation are constructed. Each of these constructions uses a pair of generalized 2-cocycles on a differential Lie superalgebra.

Designed as a postdoctoral-level reference work, this volume offers a valuable insight into new applications of supercalculus for mathematicians and physicists.

Cezar Brumă

U. KULISCH (Ed.), **PASCAL-SC — A PASCAL Extension for Scientific Computation. Information Manual and Floppy Disks**, Wiley-Teubner Series in Computer Science, B. G. Teubner, Stuttgart, John Wiley & Sons, Chichester-New York etc., V + 204 p., 1987, ISBN 3 519 02106 4 (Teubner), ISBN 0 471 91514 9 (Wiley).

PASCAL-SC is an extension of the PASCAL programming language towards providing a powerful tool and convenient facilities for solving scientific problems. The new PASCAL-SC compiler is the result of a long-term effort of a team of 16 specialized scientists — the PASCAL-SC associates — and retains the features of Standard PASCAL, ordinary PASCAL programs being naturally executed in PASCAL-SC. The floating-point arithmetic provided by PASCAL-SC is implemented not only for real floating-point numbers (type real) but also for complex numbers, intervals, vectors, and matrices over these types. The PASCAL-SC floating-point arithmetic is constructed on the basis of the theory of computer arithmetic given by U. W. Kulisch and W. L. Miranker.

The main additional features of PASCAL-SC are: a screen-oriented editor with fast syntax checking and reporting of all compilation errors; universal operator concept; functions with arbitrary result type; numerous arithmetic standard data types and standard operators; standard functions with maximum accuracy; 12-digit decimal floating-point arithmetic with maximum accuracy and directed rounding; concise and clear programs as a result of operator notation in expressions for all data types; precompiled higher arithmetic operators resulting in shorter compilation times; maximum accuracy in matrix operations; error control by means of directed rounding in interval arithmetic; computation of bounds of high quality for the solution of linear systems, eigenvalues, eigenvectors etc.; control of rounding error permits verification of mathematical facts by the computer; management of libraries; INCLUDE facility; linking of separately compiled routines; runtime error messages with reference to source lines; graphics interface for IBM color graphics; plot library; access to DOS functions.

The PASCAL-SC system is distributed on two diskettes, one marked PASCAL-SC SYSTEM, the other marked PASCAL-SC LIBRARIES, containing the following files: system boot (17 Bytes), editor (94 216 Bytes), compiler (141 471

Bytes), interpreter-runtime handler (24 769 Bytes), miscellaneous (10 216 Bytes), external arithmetic packages and libraries (153 250 Bytes), demonstration programs (149 023 Bytes). The minimal configuration of the IBM PC (or compatible) computer, provided with DOS 2.0 or higher version, contains 256 Kbytes RAM memory, two 5.25 inch diskette drives (360 Kbytes), a display.

The manual makes a detailed description of the system installation and of the PASCAL-SC demonstration package (section B). Section C gives the PASCAL-SC language description as a standard PASCAL dialect, including the syntax diagrams of the statements. Section D represents an overview on how to edit, compile and run a PASCAL-SC program. Section E presents the interface to DOS operating system and graphics. The arithmetic packages makes the description object of the section F: „built-in” arithmetic, the arithmetic operators, the precompiled packages. The problem solving routines (Section G) are contained in four libraries, for each routine being described the input and output arguments, error codes, necessary declarations in the main program. Appendix A lists the error messages.

The book, provided with the two diskettes, is a concrete and powerful tool offered to computationally oriented scientists. This tool can be mainly applied in individual research calculus but it is also an ideal instrument for teaching computer programming oriented on scientific calculi.

N. Curteanu

FRIEDMAR STOPP, *Operatorenrechnung*, Mathematik für Ingenieure, Naturwissenschaftler, Ökonomen und Landwirte, 10; 4. Auflage, BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1987, 156 S.

This book contains an account of the operational calculus based on the Laplace transform.

In the first three chapters the *theory of the Laplace transform* is expounded thoroughly, but in an essentially elementary way. In all cases, where practical, the results of the theory are illustrated by numerical examples. The last two chapters contains the *theory of Fourier and Z-transforms*.

The book concludes with tables of Laplace, Fourier and Z-transformation.

The book is not concerned exclusively with abstract and difficult theorems in pure mathematics; practical applications are given at considerable length.

The book will prove extremely valuable to Mathematicians, Research Workers, and Engineers.

Valeriu Al. Sava

D. C. CHAMPENEY, *A Handbook of Fourier Theorems*, Cambridge University Press, 1987.

The book provides a rigorous and systematic treatment of the most important concepts and results in Fourier Analysis: *the Fourier transform, the inversion theorem, the convolution theorem, the differentiation theorem*, etc. Since the background assumed does not exceed the traditional first year university course in analysis, the book is accessible to a broad class of readers including nonspecialists in mathematics. Thus, both physicists and engineers will find here a unvaluable source of references in the topic.

The author points out very clearly the dependence of the concepts introduced upon the type of the integral used. He makes a very quick and elegant introduction to the Lebesgue integral and classical measure theory and he proves the main results in Fourier Analysis first for functions belonging to the space S of L. Schwartz. Further, he extends all these to functions in L^p and even to functions lying outside L^p . Finally, he considers the case of generalized functions, i.e. distributions.

Several examples — chosen with great care — illustrate the concepts and the results included.

I. I. Vrabie

MAVROMATIS H. A., *Exercises in Quantum Mechanics*, Reidel Texts in the Mathematical Sciences, D. Reidel Publishing Company, Dordrecht-Boston-Lancaster-Tokyo, 1987, XI + 182 pp.

This book solves hundred and fourteen problems (most of which are original) of the fundamentals of non-relativistic quantum mechanics for those who are not acquainted with the subject. Most of the problems are provided with sufficiently detailed solutions. The problems are arranged in a logical sequence and with increasing difficulty. Consequently, work on the preceding problems prepares the reader for the following ones.

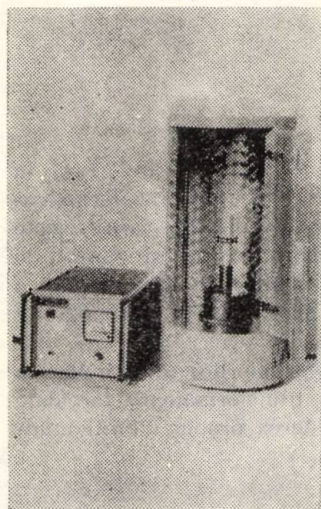
The book is divided into fourteen sections: 1. *Wilson-Sommerfeld Quantization Condition*; 2. *The Delta Function, Completeness and Closure*; 3. *Momentum Space*; 4. *Wavepackets and the Uncertainty Principle*; 5. *Uncertainty Principle and Ground-State Energies of Quantum-Mechanical Systems*; 6. *Free Particles Incident on Potentials, Time Delay, Phase Shifts and the Born Approximation*; 7. *Heisenberg Representation*; 8. *Two and Three Versus One-Dimensional Problems*; 9. *"Kramer" Type Expressions, The Virial Theorem and Generalizations*; 10. *Upper Bounds and Parity Considerations*; 11. *Perturbation Theory*; 12. *Degeneracy*; 13. *The Inverse Problem*; 14. *The Delgarno-Lewis Technique*.

The textbook will be found useful by students studying in the first years physics at universities or technical colleges.

Cezar Brumă

CENTRUL DE FIZICĂ TEHNICĂ
Str. Splai Bahlui nr. 47 IAȘI 6600
Tel. 35005, 35144, 32190

SONIFICATOR TIP BAIE



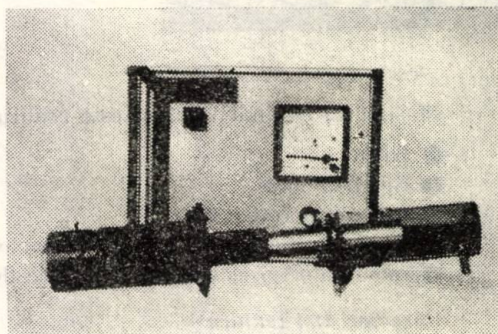
Aparatul permite tratarea materialului biologic evitându-se contaminarea preparatului cu particule metalice provenite de la ansamblul traductor ultrasonic.

Este utilizat în laboratoarele de biologie, biochimie, medicină, botanică, geologie, chimie etc.

Descriere

Sonificatorul se compune din :

- ansamblu traductor ultrasonic
- vas de tratament cu ultrasunete
- incinta fonoabsorbantă
- generator electronic



Vibrațiile ultrasonore se transmit preparatului biologic prin intermediul unei băi cu apă. Traductorul ultrasonic este de tip piezoceramic. Generatorul electronic este tranzistorizat.

Caracteristici tehnice

Tensiune de alimentare : 220 V ; 50 Hz

Putere maximă de ieșire : 200 W

Frecvența de lucru : $20 \pm 0,5$ kHz

Dimensiuni :

baia de tratament : $\varnothing = 40$ mm ; h = 70 mm

incinta fonoabsorbantă : $\varnothing = 300$ mm ; h = 600 mm

generatorul electronic : 245 × 187 × 350 mm

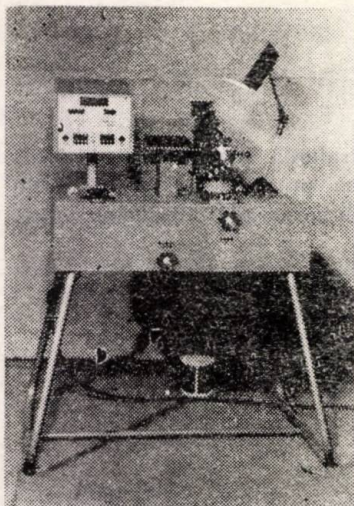
masa 25 kg

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Tel. 35005, 35144, 32190

MAȘINA DE BOBINAT TOROIDAL MBT-A 010-050



Domenii de utilizare

Mașina de bobinat toroidal se utilizează la bobinarea miezurilor toroidale ale transformatoarelor. Transformatoarele toroidale, caracterizate prin flux de scăpări foarte scăzut și curent de mers în gol mic, au căpătat în ultimii ani o vastă utilizare atât în aparatura profesională cât și în cea de larg consum.

Este destinată întreprinderilor din industria electronică care produc transformatoare toroidale și sunt interesate în creșterea productivității muncii la bobinarea acestora.

Descriere

Mașina de bobinat toroidal este realizată din :

- batiu
- cap bobinare
- mecanism rotire tor
- dispozitiv de alimentare cu conductor de bobinaj
- memorator-programator electronic de spire

Caracteristici tehnice

Dimensiunile miezului de bobinat

$O_{ext.} = \max. 65 \text{ mm}$; $O_{int.} = \min. 25 \text{ mm}$; $H_{max.} = 35 \text{ mm}$

Dimensiunile conductorului de bobinaj

$\min. 0,1 \text{ mm}$; $\max. 0,5 \text{ mm}$

Numărul de spire maxim la o singură alimentare

$\text{cond. } \varnothing 0,1 \text{ mm} - 4000 \text{ spire}$; $\text{cond. } \varnothing 0,5 \text{ mm} - 200 \text{ spire}$

Viteza de bobinare : reglabilă $0 - 150 \text{ spire/min}$

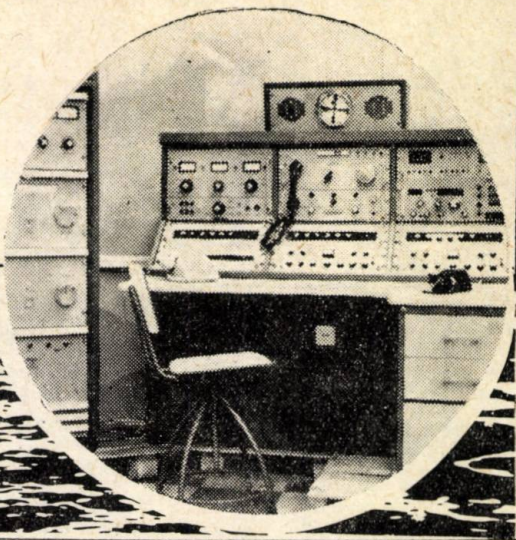
Tensiune de alimentare : 220 V , 50 Hz

Curent maxim absorbit : $1,5 \text{ A}$

Dimensiuni de gabarit : $900 \times 500 \times 1200 \text{ mm}$

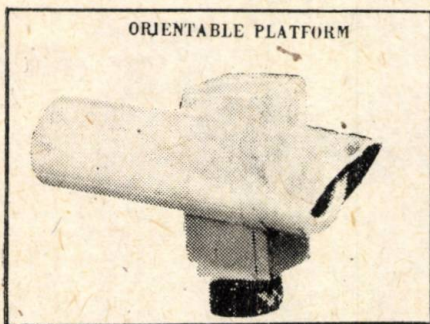
Masa netă : 100 kg

**PROFESSIONAL
RADIO-COMMUNICATION EQUIPMENT**

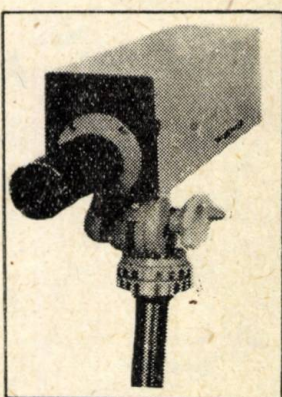


CLOSED CIRCUIT T.V. SYSTEMS

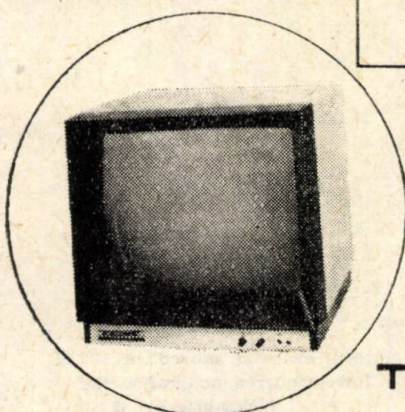
ORIENTABLE PLATFORM



**T
V
C
I**



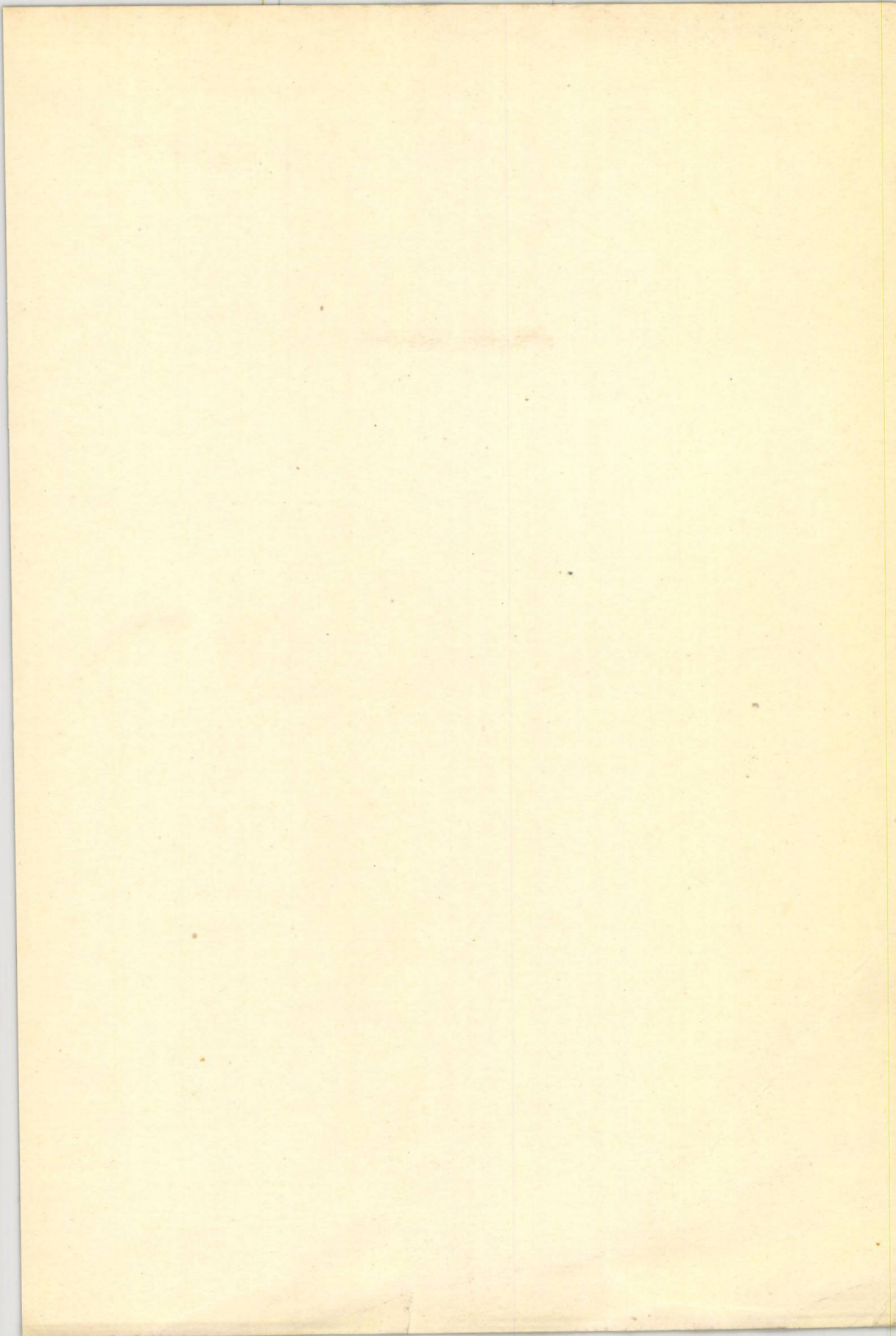
COMPACT CAMERA



TEHNOTON

MONITOR 44 cm

Tiparul executat sub cd. nr. 35 la
Intreprinderea poligrafică Iași
str. 7 Noiembrie nr. 49



40822